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Highly Symmetric Multiple Bi-Frames for Curve and Surface Multiresolution Processing

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Highly Symmetric Multiple Bi-Frames for Curve and Surface Multiresolution Processing

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Abstract

Wavelets and wavelet frames are important and useful mathematical tools in numerous applications, such as signal and image processing, and numerical analysis. Recently, the theory of wavelet frames plays an essential role in signal processing, image processing, sampling theory, harmonic analysis. However, multiwavelets and multiple frames are more flexible and have more freedom in their construction which can provide more desired properties than the scalar case, such as short compact support, orthogonality, high approximation order, and symmetry. These properties are useful in several applications, such as curve and surface noise-removing as studied in this dissertation. Thus, the study of multiwavelets and multiple frames construction has more advantages for many applications.

Recently, the construction of highly symmetric bi-frames for curve and surface multiresolution processing has been investigated. The 6-fold symmetric bi-frames, which lead to highly symmetric analysis and synthesis bi-frame algorithms, have been introduced. Moreover, these multiple bi-frame algorithms play an important role on curve and surface multiresolution processing. This dissertation is an extension of the study of construction of univariate biorthogonal wavelet frames (bi-frames for short) or dual wavelet frames of $L_2(\mathbb{R})$ with each framelet being symmetric in the scalar case. We will expand the study of biorthogonal wavelets and bi-frames construction from the scalar case to the vector case to construct biorthogonal multiwavelets and multiple bi-frames in one-dimension. In addition, we will extend the study of highly symmetric bi-frames for triangle surface multiresolution processing from the scalar case to the vector case.

More precisely, the objective of this research is to construct highly symmetric biorthogonal multiwavelets and multiple bi-frames in one and two dimensions for curve and surface multiresolution processing. It runs in parallel with the scalar case. We mainly present the methods of constructing biorthogonal multiwavelets and multiple bi-frames in both dimensions by using the idea of lifting scheme. On the whole, we discuss several topics include a brief introduction and discussion of multiwavelets theory, multiresolution analysis, scalar wavelet frames, multiple frames, and the lifting scheme. Then, we present and discuss some results of one-dimensional biorthogonal multiwavelets and multiple bi-frames for curve multiresolution processing with uniform symmetry: type I and type II along with biorthogonality, sum rule orders, vanishing moments, and uniform symmetry for both types. In addition, we present and discuss some results of two-dimensional biorthogonal multiwavelets and multiple bi-frames and the multiresolution algorithms for surface multiresolution processing. Finally, we show experimental results on curve and surface noise-removing by applying our multiple bi-frame algorithms.

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Chapter 1

Introduction

In this dissertation we will develop an extension of template-based construction of biorthogonal wavelets and wavelet bi-frames from scalar case to a vector case (multiwavelet setting) in one and two dimensions. It runs in parallel with the scalar case. We will present the method for constructing one-dimensional biorthogonal multiwavelets and multiple bi-frames with uniform symmetry: type I and type II for curve multiresolution processing. In addition, we will present the method for constructing two-dimensional biorthogonal multiwavelets and multiple bi-frames and the multiresolution algorithms.

First, we will give a brief history of frames. Frames were introduced by Duffin and Schaeffer in 1952 [34]. In 1986, Daubechies, Grossmann, and Meyer introduced wavelet frames for $L_2(\mathbb{R})$ by combining the theory of continuous wavelet transforms with the theory of frames. That was the beginning of the growth of frames [27]. Recently, the theory of frames plays an important role in signal processing, image processing, sampling theory, harmonic analysis [8, 9, 21, 25, 28, 30, 90]. Wavelet and wavelet frames in $L_2(\mathbb{R})$ have been studied extensively [2, 3, 6, 7, 8, 9, 10, 11, 19, 20, 21, 25, 26, 28, 29, 30, 34, 41, 42, 44, 45, 46, 51, 54, 55, 73, 78, 81, 82, 90] and many references therein. Bin Han and Qun Mo have generalized the construction of dual wavelet frames from the scalar refinable function to the vector case in [42]. Also, the constructions of multiwavelets and of multifilter banks have been studied in many papers such as [31, 38, 70, 71]. Multiwavelets are essential because they are more flexible and have more freedom in their construction and give us more desired properties, such as short compact support, orthogonality, high approximation order, and symmetry. These properties are useful in numerous applications.

The method of the template-based biorthogonal multiwavelets and multiple bi-frames construction is to find the parameters after solving the system of linear equations of a sum rule order of analysis and synthesis lowpass multifilters and the vanishing moments of analysis highpass multifilters for each algorithm in one and two dimensions. Then, we will obtain the corresponding analysis and synthesis matrix masks for decomposition and reconstruction algorithms and construct biorthogonal multiwavelets and multiple bi-frames. The algorithms that we use will determine the orders of sum rule and vanishing moments. However, for each algorithm, when we use the obtained parameters to construct biorthog-

onal multiwavelets and multiple bi-frames, we should select the parameters such that the multiscaling function $\tilde{\Phi}$ is smoother than Φ . Also, the analysis highpass multifilters should have higher vanishing moments. That means $\tilde{\mathbf{P}}$ has a higher sum rule order than \mathbf{P} and $\mathbf{Q}^{(\ell)}$ have higher vanishing moment order than $\tilde{\mathbf{Q}}^{(\ell)}$. By minimization (using Matlab), we can select the remaining parameters such that the multiscaling function Φ and/or its dual $\tilde{\Phi}$ have optimal smoothness. The same methodology that we use for constructing biorthogonal multiwavelets will also be used to construct multiple bi-frames.

This dissertation is organized as the following: In Chapter 2, we provide the preliminaries and notations that we use throughout this research. Specifically, we first briefly introduce and discuss the history and overview of multiwavelets, refinable function vectors, multiresolution analysis, biorthogonal wavelet, biorthogonal multiwavelets, wavelet bi-frames, multiple bi-frames, and the lifting scheme. Then, we provide sum rule order and vanishing moments of the matrix masks. Moreover, we will introduce the Sobolev smoothness estimate. We use the smoothness formula which is developed in 2003 by Jia and Jiang [62], to find the smoothness order for analysis and synthesis multiscaling functions. Thus, we consider Sobolev smoothness when we consider the smoothness of biorthogonal multiwavelets and multiple bi-frames.

In Chapter 3, we present and discuss one-dimensional (1-D for short) biorthogonal multiwavelets for curve multiresolution processing and associated multiresolution algorithm templates. The multifilter bank combines of a lowpass filter and highpass filter. In this case, we have one lowpass output and one highpass output that means $\ell = 1$. We present template-based biorthogonal multiwavelets construction: 2-step multiwavelet algorithm and 3-step multiwavelet algorithm.

In Chapter 4, we study the one-dimensional biorthogonal multiwavelet (affine) frames for curve multiresolution processing (in the vector case) as it studied in [73] (scalar case). Our idea is to extend the study in [73] from the scalar case to the vector case by following the similar steps and approaches. We develop one-dimensional multiple bi-frames and associated multiresolution algorithm templates. We obtain some results of the template-based construction of multiple bi-frames with uniform symmetry: type I. We will discuss 2-step type I multiple bi-frame algorithm, 3-step type I multiple bi-frame algorithm, and uniform symmetry. In addition, we obtain some results of the template-based construction of multiple bi-frames with uniform symmetry: type II. We will discuss 2-step type II multiple frame algorithm, 3-step type II multiple frame algorithm, and uniform symmetry. Finally, we use 1-D denoising multiple bi-frames algorithm for curve noise-removing.

In Chapter 5, we present and discuss the construction of two-dimensional (2-D for short) biorthogonal multiwavelets for surface multiresolution processing. The multifilter bank combines of a lowpass filter and three highpass filters that means $\ell = 3$. Recently, hexagonal data/image processing has essential advantages. If we compare hexagonal lattice with square lattice, we will see that hexagonal lattice has more advantages. Hexagonal

lattice has 6-fold symmetry (6 axes of symmetry) while square lattice has 4-fold symmetry. Thus, hexagonal lattice has high symmetry which will make the multifilter banks along it also have 6-fold symmetry. We presents 2-step and 3-step biorthogonal multiwavelet multiresolution algorithms.

In Chapter 6, we present and discuss the construction of two-dimensional multiple bi-frames and multiresolution algorithms. Our work will be an extension of the work of highly symmetric bi-frames for triangle surface multiresolution processing, which is presented in 2011 by Jiang and Pounds, from (the scalar case) to (the vector case). They studied the biorthogonal wavelets and bi-frames for triangular mesh-based surface multiresolution (multiscale) processing. We will show how to represent multiple bi-frame multiresolution algorithms for regular vertices as templates. Then, they are implementable. This will give a useful idea of the method of template-based multiple bi-frame construction. Moreover, we will develop 6-fold symmetry dyadic multiple bi-frames and associated templates and obtain some results of 2-step and 3-step multiple bi-frame multiresolution algorithms. In addition, we use 2-D denoising multiple bi-frame algorithm for surface noise-removing. Finally, we end this dissertation with the conclusion and future work in Chapter 7.

Chapter 2

Preliminaries

In this section we discuss the important preliminaries that we need in this research. Wavelets are important and useful mathematical tool in many applications, such as signal and image processing, numerical analysis.

In Subsection 2.1, we introduce some notations that we use throughout this research. In addition, we discuss multiwavelets theory. We introduce multiscaling function and its matrix mask. Also, multiresolution analysis (MRA) and its properties are studied in the multiwavelet setting. In Subsection 2.2, we discuss scalar wavelet frames and multiple frames and their properties. In Subsection 2.3, we introduce the lifting scheme that is a flexible and great strength method for constructing biorthogonal multifilter banks with desirable properties. By using the concept of the lifting scheme, we develop multiwavelets and multiple frame (multiple frame) algorithms that are given by several iterative steps with each steps can be represented by a symmetric template. In Subsection 2.4, we present sum rule order for vector case. In Subsection 2.5, we present vanishing moments for vector case. Finally, in Subsection 2.6, we introduce Sobolev smoothness estimate.

2.1 Multiwavelets Theory

At first, we will introduce some notations. Throughout this research, we use the **dilation factor** $d = 2$ and the **multiplicity** $r = 2$, ($r = 1$ in the scalar case). Let $L_2(\mathbb{R})$ be the Hilbert space of all integrable complex-valued functions over \mathbb{R} (the real line), equipped with

$$\| f \|^2 := \int_{\mathbb{R}} |f(x)|^2 dx,$$

and the corresponding inner product

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) \overline{g(x)} dx, \text{ for } f, g \in L_2(\mathbb{R}). \quad (2.1)$$

In this research, we use bold-faced letters to denote matrices or vectors. Let $\mathbf{k}, \boldsymbol{\omega}$ to denote elements of \mathbf{Z}^2 and \mathbb{R}^2 respectively. In general, a multi-index \mathbf{k} of \mathbf{Z}^2 and a point $\boldsymbol{\omega}$

in \mathbb{R}^2 are written as row vectors $\mathbf{k} = (k_1, k_2)$ and $\boldsymbol{\omega} = (\omega_1, \omega_2)$. We also use the notations: For a positive integer n , \mathbf{I}_n , $\mathbf{0}_n$ denote the $n \times n$ identity matrix and zero matrix respectively. Here we will use 2×2 matrices. So, \mathbf{I}_2 is 2×2 identity matrix and $\mathbf{0}_2$ is 2×2 zero matrix. Also, $\mathbf{A}^* = \overline{\mathbf{A}}^T$ denotes the complex conjugate and transpose of a matrix \mathbf{A} .

2.1.1 Refinable Function Vectors

The main differences of switching from scalar wavelets to multiwavelets (or vector case) are the scaling function ϕ and the wavelet functions ψ now are vector functions $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}^{(\ell)}$ (more than one wavelets) respectively, and the symbols (masks) are now $r \times r$ trigonometric matrix polynomials, and the recursion coefficients are $r \times r$ matrices, and so on.

In multiwavelet case [53], we let $L_2^r(\mathbb{R})$ be the set of all $r \times 1$ column vectors of functions in $L_2(\mathbb{R})$. We replace the scaling function ϕ by a function vector $\boldsymbol{\Phi}$. An $r \times 1$ scaling function vector $\boldsymbol{\Phi}$ is called a **multiscaling function** (or a vector-valued function) that is

$$\boldsymbol{\Phi}(x) := \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_r(x) \end{pmatrix} : \mathbb{R} \rightarrow \mathbb{C}^{r \times 1}, \quad (2.2)$$

where $\phi_i: \mathbb{R} \rightarrow \mathbb{C}$, $i = 1, \dots, r$ are scalar functions.

We say that a multiscaling function is a **refinable function** (or **distribution**) **vector** if it satisfies a two-scale matrix refinement equation of the form

$$\boldsymbol{\Phi} = \sum_{k \in \mathbb{Z}} \mathbf{P}_k \boldsymbol{\Phi}(2x - k), \quad (2.3)$$

where $\mathbf{P} = \{\mathbf{P}_k\}_k$ is a finitely supported sequence of $r \times r$ matrices of complex numbers on \mathbb{Z} (that is, $\mathbf{P} : \mathbb{Z} \rightarrow \mathbb{C}^{r \times r}$), and \mathbf{P} is the **matrix mask** with multiplicity r for the refinable function vector $\boldsymbol{\Phi}$, where the integer 2 is the dilation factor. When the multiplicity $r = 1$, the refinable function vector $\boldsymbol{\Phi}$ will be a scalar function and therefore we call ϕ a scalar refinable function with a scalar mask p .

We give a characterization of wavelet frames in terms of the Fourier transform, defined by

$$\hat{f}(\omega) := \int_{\mathbb{R}} f(x) e^{-ix\omega} dx, \text{ for } f \in L_1(\mathbb{R}).$$

Frequency domain methods are important for easy construction of scaling functions, regularity, symmetry, orthogonality, and biorthogonality in the multiwavelet setting. In the frequency domain, the matrix refinement equation $\boldsymbol{\Phi}$ in (2.3) is defined as

$$\hat{\boldsymbol{\Phi}}(2\omega) = \mathbf{P}(\omega) \hat{\boldsymbol{\Phi}}(\omega), \quad \omega \in \mathbb{R}, \quad (2.4)$$

The **symbol** $\mathbf{P}(\omega)$ of the sequence $\{\mathbf{P}_k\}_k$ is defined by

$$\mathbf{P}(\omega) := \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{P}_k e^{-ik\omega}, \quad \omega \in \mathbb{R}. \quad (2.5)$$

We call $\mathbf{P}(\omega)$ the mask for Φ which is $r \times r$ matrix of 2π -periodic trigonometric polynomials.

2.1.2 Multiresolution Analysis and Multiwavelets

A multiresolution analysis (MRA) is the representation method of the discrete wavelet transforms (DWT) and for the algorithm of the fast wavelet transform (FWT). Stephane Mallat and Yves Meyer first introduced multiresolution analysis (MRA). A Multiresolution Analysis allows us to decompose a signal into approximations and details. Many properties of wavelets are studied, such as orthogonality, symmetry, completeness, regularity, and approximation. When we have several scaling functions, we imagine more MRA settings. The idea of multiwavelets is to have several scaling functions and several wavelet functions. Multiwavelets give more freedom and advantages than scalar wavelets. For example, short support, orthogonality, symmetry, and vanishing moments. While we cannot have all these properties at the same time in the scalar wavelet but the multiwavelet system can have them all. Naturally, multiwavelets generalize the scalar wavelets.

Wavelet theory based on MRA (multiresolution analysis) and it is usually generated by one multiscaling function Φ and dilates and translates of only one multiwavelet functions $\Psi^{(\ell)}$. The notation of frame multiresolution analysis (FMRA) was introduced by Benedetto and Li [3, 7].

Define a subspace $\mathbf{V}_j \subset L_2(\mathbb{R})$ by

$$\mathbf{V}_j = \text{clos}_{L_2(\mathbb{R})} \langle \phi_{i,j,k} : 1 \leq i \leq r, j, k \in \mathbf{Z} \rangle, \quad (2.6)$$

For $f_i \in L_2(\mathbb{R})$, we define $f_{i,j,k}$ as the following

$$f_{i,j,k} = 2^{j/2} f_i(2^j x - k).$$

From [95, 71], a multiscaling function $\Phi(x) = [\phi_1(x), \dots, \phi_r(x)]^T$ in (2.2) generates a multiresolution analysis $\{\mathbf{V}_j\}_{j \in \mathbf{Z}}$ of $L_2(\mathbb{R})$, if $\{\mathbf{V}_j\}_{j \in \mathbf{Z}}$ defined in (2.6) satisfy the following properties:

- (1) $\dots \mathbf{V}_{-1} \subseteq \mathbf{V}_0 \subseteq \mathbf{V}_1 \dots$;
- (2) $\text{clos}_{L_2(\mathbb{R})}(\cup_{j \in \mathbf{Z}} \mathbf{V}_j) = L_2(\mathbb{R})$;
- (3) $\cap_{j \in \mathbf{Z}} \mathbf{V}_j = \{0\}$;
- (4) $f(x) \in \mathbf{V}_j \Leftrightarrow f(2x) \in \mathbf{V}_{j+1}, j \in \mathbf{Z}$;
- (5) there is a family $\{\phi_{i,j,k} : 1 \leq i \leq r, j, k \in \mathbf{Z}\}$ is a Riesz basis for \mathbf{V}_j .

Such that functions $\phi_1(x), \dots, \phi_r(x)$ are called scaling functions and they generate the MRA (\mathbf{V}_j) .

The complementary subspace of \mathbf{V}_j in \mathbf{V}_{j+1} is denoted by \mathbf{W}_j , $j \in \mathbf{Z}$. We let the multiwavelet $\Psi(x) = [\psi_1(x), \dots, \psi_r(x)]^T$, $\psi_i \in L_2(\mathbb{R})$, $i = 1, 2, \dots, r$, constitutes a Riesz basis for \mathbf{W}_j , that means \mathbf{W}_j defined as the following

$$\mathbf{W}_j = \text{clos}_{L_2(\mathbb{R})} \langle \psi_{i:j,k} : 1 \leq i \leq r, j, k \in \mathbf{Z} \rangle. \quad (2.7)$$

The family $\{\phi_{i:j,k}, \psi_{i:j,k} : 1 \leq i \leq r, j, k \in \mathbf{Z}\}$ constitutes a Riesz basis for \mathbf{V}_{j+1} , i.e.,

$$\mathbf{V}_{j+1} = \mathbf{V}_j + \mathbf{W}_j. \quad (2.8)$$

$\psi_1(x), \dots, \psi_r(x)$ are in $\mathbf{W}_0 \subset \mathbf{V}_1$, Hence there exists a sequence of matrices $\{\mathbf{Q}_k\}_{k \in \mathbf{Z}}$ such

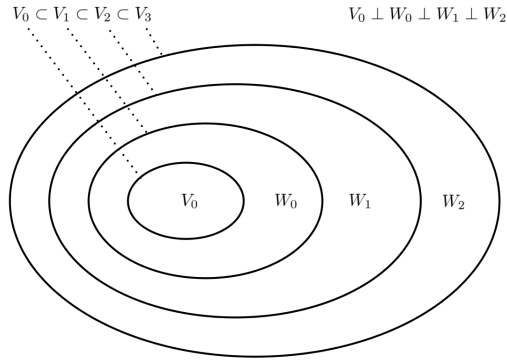


Figure 2.1: Nested subspaces

that

$$\Psi = \sum_{k \in \mathbf{Z}} \mathbf{Q}_k \Phi(2x - k), \quad (2.9)$$

and we have

$$\hat{\Psi}(2\omega) = \mathbf{Q}(\omega) \hat{\Psi}(\omega), \quad \omega \in \mathbb{R}, \quad (2.10)$$

where $\mathbf{Q}(\omega)$ is the Fourier series of the mask \mathbf{Q} , the **symbol** of the sequence $\{\mathbf{Q}_k\}_{k \in \mathbf{Z}}$, defined by

$$\mathbf{Q}(\omega) := \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{Q}_k e^{-ik\omega}, \quad \omega \in \mathbb{R}. \quad (2.11)$$

Wavelet analysis studies and applies functions ψ in $L_2(\mathbb{R})$ of the form

$$\psi_{j,k} := 2^{j/2} \psi(2^j x - k), \quad j \in \mathbf{Z}, \quad k \in \mathbf{Z}, \quad (2.12)$$

A wavelet function vector, called **multiwavelet**, defined by

$$\Psi(x) := \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_r(x) \end{pmatrix} : \mathbb{R} \rightarrow \mathbb{C}^{r \times 1}, \quad (2.13)$$

which is derived from the 2-refinable function vectors Φ , defined in (2.3). $\mathbf{Q} = \{\mathbf{Q}_k\}_{k \in \mathbf{Z}}$ is a finite sequence of $r \times r$ matrices of complex numbers on \mathbf{Z} . \mathbf{Q} is a **matrix mask** with multiplicity r for the multiwavelet function Ψ . So, Φ and Ψ together form a multiwavelet. From [37], Suppose that the $r \times r$ matrix masks $\mathbf{P}(\omega) = \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{P}_k e^{-ik\omega}$ and $\tilde{\mathbf{P}}(\omega) = \frac{1}{2} \sum_{k \in \mathbf{Z}} \tilde{\mathbf{P}}_k e^{-ik\omega}$ satisfying

$$\mathbf{P}(\omega) \tilde{\mathbf{P}}(\omega + \pi)^* + \mathbf{P}(\omega) \tilde{\mathbf{P}}(\omega + \pi)^* = \mathbf{I}_r.$$

Assume the multiscaling functions $\Phi(x) = [\phi_1(x), \dots, \phi_r(x)]^T$ and $\tilde{\Phi}(x) = [\tilde{\phi}_1(x), \dots, \tilde{\phi}_r(x)]^T$ satisfying $\Phi(x) = \sum_{k \in \mathbf{Z}} \mathbf{P}_k \Phi(2x - k)$ and $\tilde{\Phi}(x) = \sum_{k \in \mathbf{Z}} \tilde{\mathbf{P}}_k \tilde{\Phi}(2x - k)$.

Suppose the subspaces $\mathbf{V}_0(\Phi)$ and $\mathbf{V}_0(\tilde{\Phi})$ are defined as: $\mathbf{V}_0(\Phi) := \overline{\text{span}}\{\phi_l(\cdot - k) : 1 \leq l \leq r, k \in \mathbf{Z}\}$ and $\mathbf{V}_0(\tilde{\Phi}) := \overline{\text{span}}\{\tilde{\phi}_l(\cdot - k) : 1 \leq l \leq r, k \in \mathbf{Z}\}$ of $L_2(\mathbb{R})$.

Suppose the multiresolution analysis of multiplicity r with the multiscaling functions $\Phi(x)$ and $\tilde{\Phi}(x)$ are given by $\mathbf{V}_j(\Phi)$ and $\mathbf{V}_j(\tilde{\Phi})$. Thus, we can write $\Psi(x) = [\psi_1(x), \dots, \psi_r(x)]^T$ and $\tilde{\Psi}(x) = [\tilde{\psi}_1(x), \dots, \tilde{\psi}_r(x)]^T$ as:

$$\Psi(x) = \sum_{k \in \mathbf{Z}} \mathbf{Q}_k \Phi(2x - k) \text{ and } \tilde{\Psi}(x) = \sum_{k \in \mathbf{Z}} \tilde{\mathbf{Q}}_k \tilde{\Phi}(2x - k).$$

The Riesz bases of $\mathbf{V}_1(\Phi)$ and $\mathbf{V}_1(\tilde{\Phi})$ are $\{\phi_l(\cdot - k), \psi_l(\cdot - k) : 1 \leq l \leq r, k \in \mathbf{Z}\}$ and $\{\tilde{\phi}_l(\cdot - k), \tilde{\psi}_l(\cdot - k) : 1 \leq l \leq r, k \in \mathbf{Z}\}$, respectively, with $\langle \phi_l, \tilde{\phi}_{l'}(\cdot - k) \rangle = \langle \psi_l, \tilde{\psi}_{l'}(\cdot - k) \rangle = \delta_k \delta_{l-l'}$ and $\langle \phi_l, \tilde{\psi}_{l'}(\cdot - k) \rangle = \langle \phi_l, \psi_{l'}(\cdot - k) \rangle = 0$. If $\{2^{j/2} \psi_l(2^j \cdot - k) : 1 \leq l \leq r, k \in \mathbf{Z}\}$ and $\{2^{j/2} \tilde{\psi}_l(2^j \cdot - k) : 1 \leq l \leq r, k \in \mathbf{Z}\}$ generate a pair of dual Riesz bases of $L_2(\mathbb{R})$, then we said that $\Psi, \tilde{\Psi}$ form a set of biorthogonal multiwavelets.

2.2 Wavelet Frames

In Subsection 2.2.1, we introduce wavelet frames in the scalar case ($r = 1$), where r is the multiplicity. We define wavelet frames, tight wavelet frames, and a pair of dual wavelet frames in $L_2(\mathbb{R})$. In Subsection 2.2.2, we introduce wavelet frames in the vector case ($r = 2$), where r is the multiplicity. We define multiple frames, tight multiple frames, and a pair of dual multiple frames along with biorthogonality condition in $L_2(\mathbb{R})$.

2.2.1 Scalar Wavelet Frames

The theory of frames and bases play an important role as a mathematical tool in signal and image processing and it has developed recently in many studies. The interested reader can see [26, 25, 34] and references therein on the theory of frames.

let $\{\psi^{(1)}, \dots, \psi^{(L)}\} = \{\psi^{(\ell)}\}$, $l = 1, \dots, L$, be a finite set of functions in $L_2(\mathbb{R})$. The functions $\{\psi^{(\ell)}\}$ in $L_2(\mathbb{R})$ are called wavelet affine if there exist positive constants A and B such that

$$A \|f\|^2 \leq \sum_{\ell=1}^L \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbf{Z}} |\langle f, \psi_{j,k}^{(\ell)} \rangle|^2 \leq B \|f\|^2, \quad \forall f \in L_2(\mathbb{R}), \quad (2.14)$$

Thus, any frame needs both a lower bound A and an upper bound B as in (2.14). The bounds A and B are not unique. The *optimal frame bounds* are when A is the biggest possible value and B is the smallest possible value.

2.2.2 Multiple Frames

This subsection introduces the definitions of multiple frames and some properties. It runs in parallel with the scalar wavelet frames and their properties in the previous subsection.

From [46], We say that $\Psi_i^{(\ell)} = [\psi_1^{(\ell)}, \dots, \psi_r^{(\ell)}]^T$ generates a **multiple frame** in $L_2(\mathbb{R})$, $\{\psi_{i,j,k}^{(\ell)} := 2^{j/2}\psi_i^{(\ell)}(2^j x - k) : i = 1, \dots, r, j, k \in \mathbf{Z}, \ell = 1, \dots, L$, if there exist two positive constants A and B such that

$$A \|f\|_{L_2(\mathbb{R})}^2 \leq \sum_{\ell=1}^L \sum_{i=1}^r \sum_{j,k \in \mathbf{Z}} |\langle f, \psi_{i,j,k}^{(\ell)} \rangle|^2 \leq B \|f\|_{L_2(\mathbb{R})}^2, \quad \forall f \in L_2(\mathbb{R}), \quad (2.15)$$

The set $\{\psi_1^{(\ell)}, \dots, \psi_r^{(\ell)}\}$ and $\{\tilde{\psi}_1^{(\ell)}, \dots, \tilde{\psi}_r^{(\ell)}\}$ generate a **pair of dual multiple wavelet frames** in $L_2(\mathbb{R})$ if both $\{\psi_1^{(\ell)}, \dots, \psi_r^{(\ell)}\}$ and $\{\tilde{\psi}_1^{(\ell)}, \dots, \tilde{\psi}_r^{(\ell)}\}$ generate multiple wavelet frames in $L_2(\mathbb{R})$ and they satisfy

$$\langle f, g \rangle = \sum_{\ell=1}^L \sum_{i=1}^r \sum_{j,k \in \mathbf{Z}} \langle f, \psi_{i,j,k}^{(\ell)} \rangle \langle \tilde{\psi}_{j,k}^{(\ell)}, g \rangle, \quad \forall f, g \in L_2(\mathbb{R}). \quad (2.16)$$

We say that the set $\Psi_i^{(\ell)}$ generates a **(normalized) tight multiple wavelet frame** in $L_2(\mathbb{R})$ if $A = B = 1$, defines as follow

$$\sum_{\ell=1}^L \sum_{i=1}^r \sum_{j,k \in \mathbf{Z}} |\langle f, \psi_{i,j,k}^{(\ell)} \rangle|^2 = \|f\|^2. \quad (2.17)$$

In this research we let the multiplicity $r = 2$. Thus the multiscaling function is

$$\Phi(x) := \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} : \mathbb{R} \rightarrow \mathbb{C}^{2 \times 1}, \quad (2.18)$$

where ϕ is a scalar wavelet function and the multiwavelets are defined as

$$\Psi^{(\ell)}(x) := \begin{pmatrix} \psi_1^{(\ell)}(x) \\ \psi_2^{(\ell)}(x) \end{pmatrix} : \mathbb{R} \rightarrow \mathbb{C}^{2 \times 1}, \quad (2.19)$$

where $\psi^{(\ell)} : \mathbb{R} \rightarrow \mathbb{C}$, $\ell = 1, \dots, L$, $\psi^{(\ell)}$ are scalar wavelet functions.

2.3 The Lifting Scheme

The lifting scheme is a flexible and powerful method for constructing biorthogonal filter banks with desirable properties [24]. Wim Sweldens developed the lifting scheme to construct biorthogonal wavelets [92]. So, any set of compactly supported biorthogonal multiwavelets can be obtained by using the lifting scheme in vector case (multiwavelet setting). The lifting-based wavelet transform, shown in Fig.2.2, is a scheme of three operations: split, predict and update. The first operation, a signal X splits into even X_e and odd X_o parts (the lazy wavelet transform). Then, details (difference between odd elements and predict function calculated from the even elements $P(\text{even})$) will calculate by using a predictor. Predict the odd elements from the even elements outputs detail. The last operation, the average will be computed at this step, and updating the even part by using the details previously calculated. The even elements or the averages will be the input for the next repeated step of the forward lifting scheme. The reconstruction algorithm is the same as decomposition algorithm, but using the reverse process that means we first predict the data, then update it and finally merge it. Suppose Predict P and Update U are linear.

$$\begin{aligned} X_o^{new} &= X_o - PX_e; \\ X_e^{new} &= X_e + UX_o^{new} = X_e + UX_o - UPX_e. \end{aligned}$$

or in a matrix form:

$$\begin{bmatrix} X_o^{new} \\ X_e^{new} \end{bmatrix} = \begin{bmatrix} 1 & -P \\ U & 1 - UP \end{bmatrix} \begin{bmatrix} X_o \\ X_e \end{bmatrix}.$$

The lifting scheme is an important computational scheme to compute wavelet transforms. The multiwavelet lifting scheme is a flexible tool for construction of biorthogonal multiwavelets. The polyphase matrix used to derive the lifting scheme directly. We can introduce the lifting scheme via an alternative approach to wavelets called subdivision schemes which is easier to consider more general situations and applications more than wavelet filter banks.

In this research, we use the concept of the lifting scheme to design multiwavelets and multiple bi-frames. The first step, begin with the decomposition and reconstruction algorithms with symmetric templates where these algorithm templates are given by many iterative steps and these steps are given by templates. The second step, to obtain the corresponding biorthogonal multiple frame filter banks that are given by some obtained parameters. Finally, the proper selection of the parameters depends on the smoothness and vanishing moments of wavelet frames.

In the next sections, our work runs in parallel with the (scalar case) in [73], by using lifting scheme, we introduce multiple frame algorithms given by many iterative steps. Every step in the algorithms will be represented by a symmetric template.

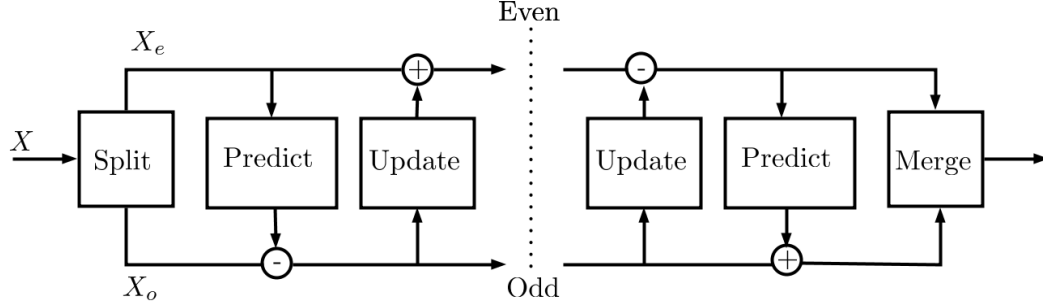


Figure 2.2: The lifting-based wavelet transform.

2.4 Sum Rule of Order of Matrix Mask \mathbf{P}

From [74, 80], for the multiscaling function $\Phi = [\phi_1, \phi_2]^T$, we assume that the associated matrix mask $\mathbf{P} = \{\mathbf{P}_k\}_{k \in \mathbf{Z}}$ is supported in the interval $[-N, N]$ for some $N > 0$. So, $\mathbf{P}_k = \mathbf{0}$ for $|k| > N$.

From [16], we assume Φ satisfies the condition of generalized partition of unity

$$\mathbf{y}_0 \sum_{k \in \mathbf{Z}} \Phi(x - k) = 1, \quad x \in \mathbb{R}, \quad \text{with} \quad \mathbf{y}_0 \in \mathbb{R}^2, \quad \mathbf{y}_0 \neq [0, 0]. \quad (2.20)$$

In this dissertation, we assume $\mathbf{y}_0 = [1, 0]$. The matrix mask $\mathbf{P} = \{\mathbf{P}_k\}_{k \in \mathbf{Z}}$ has sum rule of order 3 if there exist constant vectors $\mathbf{y}_0 = [a_1, a_2] \neq [0, 0]$, $\mathbf{y}_1 = [c_1, c_2]$, $\mathbf{y}_2 = [d_1, d_2]$, such that

$$\mathbf{y}_0 \sum_{k \in \mathbf{Z}} \mathbf{P}_{2k} = \mathbf{y}_0 \sum_{k \in \mathbf{Z}} \mathbf{P}_{2k+1} = \mathbf{y}_0. \quad (2.21)$$

$$\sum_{k \in \mathbf{Z}} (2\mathbf{y}_1 + (-2k)\mathbf{y}_0) \mathbf{P}_{2k} = \sum_{k \in \mathbf{Z}} (2\mathbf{y}_1 + (-2k - 1)\mathbf{y}_0) \mathbf{P}_{2k+1} = \mathbf{y}_1. \quad (2.22)$$

$$\begin{aligned} \sum_{k \in \mathbf{Z}} (4\mathbf{y}_2 + 4(-2k)\mathbf{y}_1 + (-2k)^2\mathbf{y}_0) \mathbf{P}_{2k} \\ = \sum_{k \in \mathbf{Z}} (4\mathbf{y}_2 + 4(-2k - 1)\mathbf{y}_1 + (-2k - 1)^2\mathbf{y}_0) \mathbf{P}_{2k+1} = \mathbf{y}_2. \end{aligned} \quad (2.23)$$

That means, if the matrix mask $\mathbf{P} = \{\mathbf{P}_k\}_{k \in \mathbf{Z}}$ satisfies (2.21), then it has sum rule of order 1, (2.22) for sum rule of order 2, and (2.21) for sum rule of order 3. Sum rule order of the matrix mask \mathbf{P} implies the accuracy order of the associated Φ . Namely, with

$$\mathbf{y}_0(j) = \mathbf{y}_0, \quad \mathbf{y}_1(j) = \mathbf{y}_1 + j\mathbf{y}_0, \quad \mathbf{y}_2(j) = \mathbf{y}_2 + 2j\mathbf{y}_1 + j^2\mathbf{y}_0, \quad j \in \mathbf{Z},$$

From [74] and the references therein, for $n = 0, 1, 2$, we have

$$\mathbf{x}^n = \sum_{j \in \mathbf{Z}} \mathbf{y}_n(j) \Phi(x - j), \quad x \in \mathbb{R}.$$

Thus, when the matrix mask \mathbf{P} has sum rule of order 3, the polynomial of degree ≤ 2 can be reproduced by $\phi_1(x-j), \phi_2(x-j), j \in \mathbf{Z}$.

Since the mask $\mathbf{P}(\omega)$ has sum rule of order 1, which is the basic sum rule, then we have the following

$$\mathbf{y}_0 \mathbf{P}(0) = \mathbf{y}_0$$

$$\mathbf{y}_0 \mathbf{P}(\pi) = [0, 0],$$

where $\mathbf{y}_0 = [1, 0]$.

The mask $\tilde{\mathbf{P}}(\omega)$ has sum rule of order 1, the basic sum rule, then we have the following

$$\tilde{\mathbf{y}}_0 \tilde{\mathbf{P}}(0) = \tilde{\mathbf{y}}_0$$

$$\tilde{\mathbf{y}}_0 \tilde{\mathbf{P}}(\pi) = [0, 0],$$

A matrix is said to satisfy **Condition E** if

- Its spectral radius (the absolute values of its eigenvectors) is 1,
- 1 is a simple eigenvalue,
- 1 is the unique eigenvalue on the unit circle.

2.5 Vanishing Moment of Order p

In multiwavelet case, vanishing moments order of $\Psi^{(\ell)} = [\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(L)}]^T, \ell = 1, \dots, L$ is an important property. For multiwavelet we have one highpass multifilter $\ell = 1$. Unlike multiple bi-frames we have more than one highpass multifilter.

If $\mathbf{Q}^{(\ell)}(\omega), \ell = 1, \dots, L$ has vanishing moments of order one and $\Psi^{(\ell)}$ is the compactly supported function defined by $\hat{\Psi}^{(\ell)}(2\omega) = \mathbf{Q}^{(\ell)}(\omega) \hat{\Phi}(\omega)$, where Φ is compactly supported function in $L_2(\mathbb{R})$, then $\Psi^{(\ell)}$ has vanishing moments of order one:

$$\hat{\Psi}^{(\ell)}(0) = \mathbf{Q}^{(\ell)}(0) \hat{\Phi}(0) = [0, 0]^T, \ell = 1, \dots, L, \quad (2.24)$$

We have assumed Φ satisfies the condition of generalized partition of unity (2.20). So, let $\mathbf{c}_0 = [1, 0]$. For $\mathbf{Q}^{(\ell)}(\omega), \ell = 1, \dots, L$, when it is used as the analysis highpass filter, we say it has vanishing moments of order one if

$$\mathbf{c}_0 \mathbf{Q}^{(\ell)}(0)^T = [0, 0].$$

2.6 Sobolev Smoothness Estimate

In this dissertation we will use the smoothness formula provided in [62], which is developed by Jia and Jiang. We have used the Matlab routines developed for Sobolev smoothness computations of refinable function vectors [68]. The Sobolev space denote by $W_2^\lambda(\mathbb{R}^s)$ of all functions $f \in (\mathbb{R}^s)$ such that

$$\int_{\mathbb{R}^s} |\hat{f}(\xi)|^2 (1 + |\xi|^\lambda)^2 d\xi < \infty.$$

In one dimension, let Φ be an 2×1 function vector in $L_2(\mathbb{R})^2$, $s = 1$. Assume Φ satisfies the refinement equation (2.3) with the 2×2 matrix mask \mathbf{P} and the dilation factor is $d = 2$. The mask \mathbf{P} has optimal sum rule order p . Let $\lambda_j, j = 1, 2$ be the eigenvalues of $\mathbf{P}(0)$ and by assuming $\lambda_1 = 1$ and $|\lambda_2| < 1$, $\mathbf{P}(\omega)$ is the Fourier series of the mask \mathbf{P} , the **symbole** of the matrix mask $\{\mathbf{P}_k\}$, defined in (2.5). Thus, $\mathbf{P}(0) := \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{P}_k$. Let \mathbf{b} supported on $[-N, N] \cap \mathbf{Z}$ and $\mathbf{b}_k = \frac{1}{2} \sum_{n \in \mathbf{Z}} \bar{\mathbf{P}}_n \otimes \mathbf{P}_{k+n}$, $k, n \in \mathbf{Z}$. Let the transition operator matrix $\mathbf{T}_\mathbf{P} := [\mathbf{b}_{2k-n}]_{k,n \in [-N, N]}$, $(4 \times (2N + 1)) \times (4 \times (2N + 1))$ matrix. Let

$$\bar{E}_2 := \{\lambda_2 2^{-\mu}, \bar{\lambda}_2 2^{-\mu}, 2^{-\beta} : \mu < p, \beta < 2p\}.$$

where $\mu, \beta \in \mathbb{N}_0$ which is the set of nonnegative integers. We define

$$E_2 := \text{spec}(\mathbf{T}_\mathbf{P}) \setminus \bar{E}_2$$

Let define

$$\rho_2 := \max\{|\lambda| : \lambda \in E_2\}.$$

Thus, we have

$$\lambda(\Phi) = -(\log_2 \rho_2)/2, \quad (2.25)$$

where the critical exponent $\lambda(\Phi)$ defined as

$$\lambda(\Phi) := \sup\{\lambda : \phi_j \in W_2^\lambda(\mathbb{R}), j = 1, 2\}.$$

We will paraphrase the theory of the Sobolev smoothness estimate for two dimensional case [62]. Let $\mathbf{M} = 2\mathbf{I}_2$, $d := |\det(\mathbf{M})| = 4$, (dilation matrix), $s = 2, r = 2$. \mathbf{M} is similar to a diagonal matrix $\text{diag}(2, 2)$. The spectrum of the matrix \mathbf{M} is $\text{spec}(\mathbf{M}) = \{2, 2\}$. Let 2-tuple $\mu = (\mu_1, \mu_2)$, $\beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$, where \mathbb{N}_0^2 the set of nonnegative integers. μ and β are called multi-indices. So, $2^{-\mu} = 2^{-\mu_1} \cdot 2^{-\mu_2}$, $2^{-\beta} = 2^{-\beta_1} \cdot 2^{-\beta_2}$ and the lengths of μ and β are $|\mu| := \mu_1 + \mu_2$, $|\beta| := \beta_1 + \beta_2$. Let $\lambda_j, j = 1, 2$ be the eigenvalues of $\mathbf{P}(0, 0)$ and by assuming $\lambda_1 = 1$ and $|\lambda_2| < 1$, $\mathbf{P}(\omega)$ is the Fourier series of the mask \mathbf{P} which has optimal sum rule order p , the **symbole** of the matrix mask $\{\mathbf{P}_k\}$, defined by

$$\mathbf{P}(\omega) := \frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^2} \mathbf{P}_k e^{-i\mathbf{k} \cdot \omega}, \quad \omega \in \mathbb{R}^2.$$

Thus, $\mathbf{P}(0, 0) := \frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^2} \mathbf{P}_{\mathbf{k}}$. Let \mathbf{b} supported on $[-N, N]^2 \cap \mathbf{Z}^2$ and $\mathbf{b}_{\mathbf{k}} = \frac{1}{4} \sum_{\mathbf{n} \in \mathbf{Z}^2} \bar{\mathbf{P}}_{\mathbf{n}} \otimes \mathbf{P}_{\mathbf{k}+\mathbf{n}}$, $\mathbf{k}, \mathbf{n} \in \mathbf{Z}^2$. Let the transition operator matrix $\mathbf{T}_{\mathbf{P}} := \left[\mathbf{b}_{2\mathbf{k}-\mathbf{n}} \right]_{\mathbf{k}, \mathbf{n} \in [-N, N]^2}$, $(4 \times (2N + 1)^2) \times (4 \times (2N + 1)^2)$ matrix. Let

$$\bar{E}_2 := \{\lambda_2 2^{-\mu}, \bar{\lambda}_2 2^{-\mu}, 2^{-\beta} : |\mu| < p, |\beta| < 2p\}.$$

We define

$$E_2 := \text{spec}(\mathbf{T}_{\mathbf{P}}) \setminus \bar{E}_2$$

Let define

$$\rho_2 := \max\{|\lambda| : \lambda \in E_2\}.$$

Thus, we have

$$\lambda(\Phi) = -(\log_4 \rho_2), \tag{2.26}$$

where the critical exponent $\lambda(\Phi)$ defined as

$$\lambda(\Phi) := \sup\{\lambda : \phi_j \in W_2^\lambda(\mathbb{R}^2), j = 1, 2\}.$$

Chapter 3

One-dimensional Biorthogonal Multiwavelets and Associated Multiresolution Algorithm Templates

3.1 Introduction

In [73], Jiang has presented the construction of symmetric biorthogonal wavelet in the scalar case. By following the work in [73], we present the construction of biorthogonal wavelet in multiwavelet case (vector case).

The multifilter bank combines of a lowpass filter and highpass filter. In this case, we have one lowpass output and one highpass output that means $\ell = 1$. In section 3.2, we provide some new results on biorthogonal multiwavelets and associated multiresolution algorithm templates. We present 2-step and 3-step biorthogonal multiwavelet algorithms.

The FIR multifilter banks $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$ are said to be biorthogonal multifilter banks if they satisfy the biorthogonality conditions:

$$\begin{cases} \mathbf{P}(\omega) \tilde{\mathbf{P}}(\omega)^* + \mathbf{P}(\omega + \pi) \tilde{\mathbf{P}}(\omega + \pi)^* = \mathbf{I}_2, \\ \mathbf{P}(\omega) \tilde{\mathbf{Q}}(\omega)^* + \mathbf{P}(\omega + \pi) \tilde{\mathbf{Q}}(\omega + \pi)^* = \mathbf{0}_2, \\ \mathbf{Q}(\omega) \tilde{\mathbf{P}}(\omega)^* + \mathbf{Q}(\omega + \pi) \tilde{\mathbf{P}}(\omega + \pi)^* = \mathbf{0}_2, \\ \mathbf{Q}(\omega) \tilde{\mathbf{Q}}(\omega)^* + \mathbf{Q}(\omega + \pi) \tilde{\mathbf{Q}}(\omega + \pi)^* = \mathbf{I}_2, \end{cases} \quad \omega \in \mathbb{R} \quad (3.1)$$

$\mathbf{A}^* = \overline{\mathbf{A}}^T$ denotes the transpose of the complex conjugate of \mathbf{A} and \mathbf{A}^T denotes the transpose of \mathbf{A} .

Let $\mathbf{M}_{\mathbf{P}, \mathbf{Q}}(\omega)$ be the analysis modulation matrix and $\mathbf{M}_{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}}(\omega)$ be the synthesis modu-

lation matrix of $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$, respectively, defined by

$$\mathbf{M}_{\mathbf{P}, \mathbf{Q}} := \begin{bmatrix} \mathbf{P}(\omega) & \mathbf{P}(\omega + \pi) \\ \mathbf{Q}(\omega) & \mathbf{Q}(\omega + \pi) \end{bmatrix}_{4 \times 4}, \quad (3.2)$$

$$\mathbf{M}_{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}} := \begin{bmatrix} \tilde{\mathbf{P}}(\omega) & \tilde{\mathbf{P}}(\omega + \pi) \\ \tilde{\mathbf{Q}}(\omega) & \tilde{\mathbf{Q}}(\omega + \pi) \end{bmatrix}_{4 \times 4}. \quad (3.3)$$

Then (3.1) is equivalent to the following single equation:

$$\mathbf{M}_{\mathbf{P}, \mathbf{Q}}(\omega) \mathbf{M}_{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}}(\omega)^* = \mathbf{I}_4, \quad \omega \in \mathbb{R}, \quad (3.4)$$

where \mathbf{M}^* denotes the complex conjugate and transpose of a matrix \mathbf{M} .

From, [47], the perfect reconstruction condition is equivalent to showing

$$\mathbf{M}_{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}}(\omega)^* \mathbf{M}_{\mathbf{P}, \mathbf{Q}}(\omega) = \mathbf{I}_4, \quad \omega \in \mathbb{R}, \quad (3.5)$$

Since $\mathbf{M}_{\mathbf{P}, \mathbf{Q}}(\omega)$ and $\mathbf{M}_{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}}(\omega)$ are square invertible matrices, the conditions specified by equations (3.4) and (3.5) are equivalent. Thus, if matrix polynomials $\mathbf{M}_{\mathbf{P}, \mathbf{Q}}(\omega)$ and $\mathbf{M}_{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}}(\omega)$ are constructed from compactly supported biorthogonal scaling vectors $\Phi, \tilde{\Phi}$ and wavelet vectors $\Psi, \tilde{\Psi}$, then we notice that the biorthogonality conditions are equivalent to the perfect reconstruction conditions.

From [47, 88], a multifilter bank gives perfect reconstruction when

$$\begin{cases} \tilde{\mathbf{P}}(\omega)^* \mathbf{P}(\omega) + \tilde{\mathbf{Q}}(\omega)^* \mathbf{Q}(\omega) = \mathbf{I}_2, \\ \tilde{\mathbf{P}}(\omega)^* \mathbf{P}(\omega + \pi) + \tilde{\mathbf{Q}}(\omega)^* \mathbf{Q}(\omega + \pi) = \mathbf{0}_2. \end{cases} \quad (3.6)$$

These two conditions involve the modulation matrix $\mathbf{M}_{\mathbf{P}, \mathbf{Q}}(\omega)$:

$$[\tilde{\mathbf{P}}(\omega)^* \quad \tilde{\mathbf{Q}}(\omega)^*]_{2 \times 4} \times \begin{bmatrix} \mathbf{P}(\omega) & \mathbf{P}(\omega + \pi) \\ \mathbf{Q}(\omega) & \mathbf{Q}(\omega + \pi) \end{bmatrix}_{4 \times 4} = [\mathbf{I}_2 \quad \mathbf{0}_2]_{2 \times 4}. \quad (3.7)$$

3.2 Biorthogonal Multiwavelets and Associated Multiresolution Algorithm Templates

When we use a multifilter bank $\{\mathbf{P}, \mathbf{Q}\}$ as the analysis multifilter bank, the multiresolution decomposition algorithm for the input data $\{\mathbf{c}_k\}$ is

$$\tilde{\mathbf{c}}_n = \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{c}_k \mathbf{P}_{k-2n}^T, \quad \mathbf{d}_n = \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{c}_k \mathbf{Q}_{k-2n}^T, \quad (3.8)$$

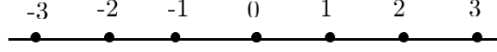


Figure 3.1: Indices for the nodes

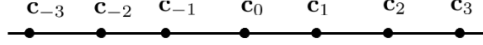


Figure 3.2: Indices for the data \mathbf{c}_k

where $\tilde{\mathbf{c}}_n$, \mathbf{c}_k and \mathbf{d}_n are 1×2 “row-vectors” and the masks \mathbf{P}_{k-2n} and \mathbf{Q}_{k-2n} are 2×2 matrices.

If an FIR multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$ is biorthogonal to $\{\mathbf{P}, \mathbf{Q}\}$, then the input data $\{\mathbf{c}_k\}$ can be recovered from $\tilde{\mathbf{c}}_n$ and \mathbf{d}_n by the multiresolution reconstruction algorithm:

$$\mathbf{c}'_k = \sum_{n \in \mathbf{Z}} \tilde{\mathbf{c}}_n \tilde{\mathbf{P}}_{k-2n} + \sum_{n \in \mathbf{Z}} \mathbf{d}_n \tilde{\mathbf{Q}}_{k-2n}, \quad k \in \mathbf{Z}, \quad (3.9)$$

where a multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$ is called the synthesis multifilter bank. $\{\tilde{\mathbf{c}}_k\}_k$ is called the lowpass output and $\{\mathbf{d}_k\}_k$ is called the highpass output of $\{\mathbf{c}_k\}_k$. $\{\tilde{\mathbf{c}}_k\}_k$ is called the “approximation” of $\{\mathbf{c}_k\}_k$ and $\{\mathbf{d}_k\}_k$ is the “detail” of $\{\mathbf{c}_k\}_k$.

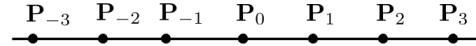


Figure 3.3: Indices for the lowpass multifilter coefficients \mathbf{P}_k

Theorem 1. *If $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$ are biorthogonal multifilter banks, then the input data $\{\mathbf{c}_k\}_k$ can be recovered from its approximation $\{\tilde{\mathbf{c}}_k\}_k$ and detail $\{\mathbf{d}_k\}_k$, namely the input data $\{\mathbf{c}_k\}_k$ is exactly $\{\mathbf{c}'_k\}_k$.*

Proof: Let $\tilde{\mathbf{c}}(2\omega)$, $\mathbf{d}(2\omega)$, $\mathbf{c}(\omega)$, and $\mathbf{c}'(\omega)$ denote the Z-transforms of $\tilde{\mathbf{c}}_n$, \mathbf{d}_n , \mathbf{c}_k , and \mathbf{c}'_k , respectively. The Z-transforms $\mathbf{c}(\omega)$ of $\mathbf{c}_k = \{\mathbf{c}_k\}_k$ is $\mathbf{c}(\omega) = \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{c}_k e^{-ik\omega}$. In the frequency domain, we have

$$\tilde{\mathbf{c}}(2\omega) = \frac{1}{2} \left(\mathbf{c}(\omega) \overline{\mathbf{P}(\omega)}^T + \mathbf{c}(\omega + \pi) \overline{\mathbf{P}(\omega + \pi)}^T \right),$$

$$\mathbf{d}(2\omega) = \frac{1}{2} \left(\mathbf{c}(\omega) \overline{\mathbf{Q}(\omega)}^T + \mathbf{c}(\omega + \pi) \overline{\mathbf{Q}(\omega + \pi)}^T \right),$$

and

$$\mathbf{c}'(\omega) = 2\left(\tilde{\mathbf{c}}(2\omega)\tilde{\mathbf{P}}(\omega) + \mathbf{d}(2\omega)\tilde{\mathbf{Q}}(\omega)\right).$$

Suppose that the multifilter banks $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$ are biorthogonal. By plugging $\tilde{\mathbf{c}}(2\omega), \mathbf{d}(2\omega)$ into $\mathbf{c}'(\omega)$ and if the conditions in (3.6) hold. Then, $\mathbf{c}'(\omega) = \mathbf{c}(\omega)$. Hence, the biorthogonality of $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$ implies that the original data can be recovered from the lowpass and highpass outputs by the multiwavelet reconstruction algorithms.

We will describe the association. We will associate the lowpass and highpass output to the nodes of \mathbf{Z} that means we represent multiresolution algorithms by templates. Associate $\tilde{\mathbf{v}}_k$ with an even node $2k$ and $\tilde{\mathbf{e}}_k$ with an odd node $2k + 1$. For initial data $\{\mathbf{c}_k\}$, denotes

$$\mathbf{v}_k = \mathbf{c}_{2k}, \mathbf{e}_k = \mathbf{c}_{2k+1}, k \in \mathbf{Z}, \quad (3.10)$$

and

$$\tilde{\mathbf{v}}_k = \tilde{\mathbf{c}}_k, \tilde{\mathbf{e}}_k = \mathbf{d}_k, k \in \mathbf{Z}, \quad (3.11)$$

where $\mathbf{v}_k, \mathbf{e}_k, \tilde{\mathbf{v}}_k$, and $\tilde{\mathbf{e}}_k$ are 1×2 row-vectors.

3.2.1 2-Step Multiwavelet Algorithm

let us consider a 2-step multiwavelet multiresolution algorithm. The decomposition algorithm of this 2-step algorithm is given in (3.12)-(3.13). The decomposition Step1 given in (3.12), we obtain lowpass output $\tilde{\mathbf{v}}$ by replacing each \mathbf{v} associated with an even node $2k$ by $\tilde{\mathbf{v}}$. Then, in Step 2, we use the obtained $\tilde{\mathbf{v}}$ to obtain the highpass output $\tilde{\mathbf{e}}$ associated with odd nodes $2k + 1$ given in (3.13).

2-step Decomposition Algorithm:

$$\text{Step 1. } \tilde{\mathbf{v}} = \{\mathbf{v} - (\mathbf{e}_{-1} + \mathbf{e}_0)\mathbf{D}\}\mathbf{B}^{-1}; \quad (3.12)$$

$$\text{Step 2. } \tilde{\mathbf{e}} = \mathbf{e} - (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1)\mathbf{U}. \quad (3.13)$$

2-step Reconstruction Algorithm:

$$\text{Step 1. } \mathbf{e} = \tilde{\mathbf{e}} + (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1)\mathbf{U}; \quad (3.14)$$

$$\text{Step 2. } \mathbf{v} = \tilde{\mathbf{v}}\mathbf{B} + (\mathbf{e}_{-1} + \mathbf{e}_0)\mathbf{D}, \quad (3.15)$$

where \mathbf{B}, \mathbf{D} , and \mathbf{U} are 2×2 matrices and the entries of these matrices are some constants in \mathbb{R} .

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \text{ and } \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix},$$

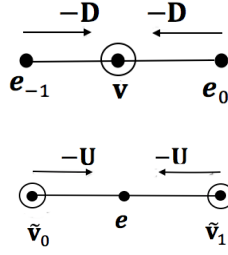


Figure 3.4: Top: Decomposition Step 1; bottom: Decomposition Step 2.

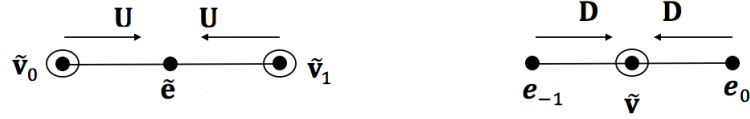


Figure 3.5: Left: Reconstruction Step 1; Right: Reconstruction Step 2.

are 2×2 matrices where the entries of these square matrices are some constants in \mathbb{R} .

The reconstruction algorithm is the backward algorithm of the decomposition algorithm. The multiresolution reconstruction algorithm is given in (3.14) and (3.15), where the matrices are the same 2×2 matrices in the decomposition algorithm.

In Step 1, we replace $\tilde{\mathbf{e}}$ of the highpass output by \mathbf{e} , the original data \mathbf{c}_{2k+1} associated with odd nodes, given in (3.14). Then in Step 2, with the obtained \mathbf{e} in Step 1, we update $\tilde{\mathbf{v}}$ of the lowpass output by \mathbf{v} by formula given in (3.15). In this step we recover original data \mathbf{c}_{2k} associated with even nodes. So the inputs here are $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{e}}$ and the outputs are \mathbf{e} , \mathbf{v} .

Now, let us obtain the corresponding biorthogonal multifilter banks (in the vector case) $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$ to this 2-step algorithm. Then, we can study the properties of the corresponding multiwavelets to choose the parameters.

From (3.11) and (3.12), we have

$$\tilde{\mathbf{c}}_n = \tilde{\mathbf{v}}_n = \{\mathbf{v}_n - (\mathbf{e}_{n-1} + \mathbf{e}_n) \mathbf{D}\} \mathbf{B}^{-1} = \{\mathbf{c}_{2n} - (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+1}) \mathbf{D}\} \mathbf{B}^{-1}. \quad (3.16)$$

From (3.11) and (3.13) we have

$$\begin{aligned} \mathbf{d}_n = \tilde{\mathbf{e}}_n &= \mathbf{e}_n - (\tilde{\mathbf{v}}_n + \tilde{\mathbf{v}}_{n+1}) \mathbf{U} \\ &= \mathbf{c}_{2n+1} - (\{\mathbf{c}_{2n} - (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+1}) \mathbf{D}\} \mathbf{B}^{-1} + \{\mathbf{c}_{2n+2} - (\mathbf{c}_{2n+1} + \mathbf{c}_{2n+3}) \mathbf{D}\} \mathbf{B}^{-1}) \mathbf{U} \\ &= \mathbf{c}_{2n+1} (\mathbf{I}_2 + 2\mathbf{D}\mathbf{B}^{-1}\mathbf{U}) - (\mathbf{c}_{2n} + \mathbf{c}_{2n+2}) \mathbf{B}^{-1}\mathbf{U} + (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+3}) \mathbf{D}\mathbf{B}^{-1}\mathbf{U}. \end{aligned} \quad (3.17)$$

We get the nonzero coefficients \mathbf{P}_k , \mathbf{Q}_k of $\mathbf{P}(\omega)$, $\mathbf{Q}(\omega)$ by comparing (3.16) and (3.17)

with (3.8):

$$\begin{aligned}\mathbf{P}_0 &= 2\mathbf{B}^{-T}, \mathbf{P}_{-1} = \mathbf{P}_1 = -2\mathbf{B}^{-T}\mathbf{D}^T; \\ \mathbf{Q}_0 &= \mathbf{Q}_2 = -2\mathbf{U}^T\mathbf{B}^{-T}, \mathbf{Q}_1 = 2(\mathbf{I}_2 + 2\mathbf{U}^T\mathbf{B}^{-T}\mathbf{D}^T), \mathbf{Q}_{-1} = \mathbf{Q}_3 = 2\mathbf{U}^T\mathbf{B}^{-T}\mathbf{D}^T\end{aligned}\quad (3.18)$$

Throughout the research we use \mathbf{B}^{-T} to denotes the transpose of the inverse of the matrix \mathbf{B} . Therefore, the analysis multifilter bank \mathbf{P}, \mathbf{Q} is

$$\mathbf{P}(\omega) = \frac{1}{2}(2\mathbf{B}^{-T} - 2(e^{i\omega} + e^{-i\omega})\mathbf{B}^{-T}\mathbf{D}^T); \quad (3.19)$$

$$\begin{aligned}\mathbf{Q}(\omega) &= \frac{1}{2}(e^{-i\omega}(2\mathbf{I}_2 + 4\mathbf{U}^T\mathbf{B}^{-T}\mathbf{D}^T) - 2(1 + e^{-2i\omega})\mathbf{U}^T\mathbf{B}^{-T} \\ &\quad + 2(e^{i\omega} + e^{-3i\omega})\mathbf{U}^T\mathbf{B}^{-T}\mathbf{D}^T).\end{aligned}\quad (3.20)$$

We will obtain the synthesis multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$ from (3.10) and (3.14) we have

$$\begin{aligned}\mathbf{c}_{2k+1} &= \mathbf{e}_k = \tilde{\mathbf{e}}_k + (\tilde{\mathbf{v}}_k + \tilde{\mathbf{v}}_{k+1})\mathbf{U} \\ &= \mathbf{d}_k + (\tilde{\mathbf{c}}_k + \tilde{\mathbf{c}}_{k+1})\mathbf{U}.\end{aligned}\quad (3.21)$$

From (3.10) and (3.15)

$$\begin{aligned}\mathbf{c}_{2k} &= \mathbf{v}_k = \tilde{\mathbf{v}}_k\mathbf{B} + (\mathbf{e}_{k-1} + \mathbf{e}_k)\mathbf{D} \\ &= \tilde{\mathbf{c}}_k\mathbf{B} + \{(\mathbf{d}_{k-1} + (\tilde{\mathbf{c}}_{k-1} + \tilde{\mathbf{c}}_k)\mathbf{U} + \mathbf{d}_k + (\tilde{\mathbf{c}}_k + \tilde{\mathbf{c}}_{k+1})\mathbf{U})\mathbf{D} \\ &= \tilde{\mathbf{c}}_k(\mathbf{B} + 2\mathbf{U}\mathbf{D}) + (\tilde{\mathbf{c}}_{k-1} + \tilde{\mathbf{c}}_{k+1})\mathbf{U}\mathbf{D} + (\mathbf{d}_{k-1} + \mathbf{d}_k)\mathbf{D}.\end{aligned}\quad (3.22)$$

By comparing (3.9) and (3.21) for odd k , we have that the nonzero coefficients $\mathbf{P}_{2k+1}, \mathbf{Q}_{2k+1}$ with odd indices $2k+1$ are

$$\begin{aligned}\tilde{\mathbf{P}}_1 &= \tilde{\mathbf{P}}_{-1} = \mathbf{U}; \\ \tilde{\mathbf{Q}}_1 &= \mathbf{I}_2\end{aligned}\quad (3.23)$$

where the other coefficients are zero.

Now, we compare (3.22) and (4.11) for even k , we have that the nonzero coefficients $\mathbf{P}_{2k}, \mathbf{Q}_{2k}$ with even indices $2k$ are

$$\begin{aligned}\tilde{\mathbf{P}}_0 &= \mathbf{B} + 2\mathbf{U}\mathbf{D}, \tilde{\mathbf{P}}_2 = \tilde{\mathbf{P}}_{-2} = \mathbf{U}\mathbf{D}; \\ \tilde{\mathbf{Q}}_0 &= \tilde{\mathbf{Q}}_2 = \mathbf{D};\end{aligned}\quad (3.24)$$

Hence the synthesis multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$ is

$$\tilde{\mathbf{P}}(\omega) = \frac{1}{2}(\mathbf{B} + 2\mathbf{U}\mathbf{D} + (e^{i\omega} + e^{-i\omega})\mathbf{U} + (e^{2i\omega} + e^{-2i\omega})(\mathbf{U}\mathbf{D})); \quad (3.25)$$

$$\tilde{\mathbf{Q}}(\omega) = \frac{1}{2}(e^{-i\omega} + (1 + e^{-2i\omega})\mathbf{D}). \quad (3.26)$$

Denote

$$\begin{cases} \mathbf{A}_1(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 \\ -(1 + e^{-i\omega})\mathbf{U}^T & \mathbf{I}_2 \end{bmatrix}_{4 \times 4}, \\ \mathbf{A}_0(\omega) = \begin{bmatrix} \mathbf{B}^{-T} & -(1 + e^{i\omega})\mathbf{B}^{-T}\mathbf{D}^T \\ \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{4 \times 4}, \end{cases} \quad (3.27)$$

and $\tilde{\mathbf{A}}_1(\omega) = (\mathbf{A}_1(\omega)^{-1})^*$ and $\tilde{\mathbf{A}}_0(\omega) = (\mathbf{A}_0(\omega)^{-1})^*$ are given by

$$\begin{cases} \tilde{\mathbf{A}}_1(\omega) = \begin{bmatrix} \mathbf{I}_2 & (1 + e^{i\omega})\mathbf{U} \\ \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{4 \times 4}, \\ \tilde{\mathbf{A}}_0(\omega) = \begin{bmatrix} \mathbf{B} & \mathbf{0}_2 \\ (1 + e^{-i\omega})\mathbf{D} & \mathbf{I}_2 \end{bmatrix}_{4 \times 4}. \end{cases} \quad (3.28)$$

Then the multifilter banks $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$ corresponding to this 2-step algorithm can be written as

$$\begin{bmatrix} \mathbf{P}(\omega) \\ \mathbf{Q}(\omega) \end{bmatrix}_{4 \times 2} = \mathbf{A}_1(2\omega)\mathbf{A}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}_{4 \times 2}, \quad (3.29)$$

and

$$\begin{bmatrix} \tilde{\mathbf{P}}(\omega) \\ \tilde{\mathbf{Q}}(\omega) \end{bmatrix}_{4 \times 2} = \frac{1}{2}\tilde{\mathbf{A}}_1(2\omega)\tilde{\mathbf{A}}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}_{4 \times 2}. \quad (3.30)$$

After solving the system of linear equations for a sum rule order three of $\tilde{\mathbf{P}}$, for a sum rule order one of \mathbf{P} , and for a vanishing moment order one of \mathbf{Q} , we can find the parameters. we have one free parameter which is g and the other parameters are

$$\begin{aligned} b_{11} &= 2, \quad b_{12} = 0, \quad b_{21} = 0, \quad b_{22} = \frac{2}{9}, \quad d_{11} = \frac{-1}{2}, \quad d_{12} = 0, \quad d_{21} = 0, \quad d_{22} = \frac{1}{18}, \\ h &= \frac{1}{6}, \quad u_{11} = \frac{1}{2}, \quad u_{12} = 0, \quad u_{21} = \frac{-3}{16g}, \quad u_{22} = \frac{1}{8}. \end{aligned}$$

If we choose $g = \frac{1624}{263}$ and by using the smoothness formula, we can obtain $\tilde{\Phi} \in W^{0.4408}$ which is supported on $[-2, 2]$, but Φ is not in $L_2(\mathbb{R})$ and we cannot find it in $L_2(\mathbb{R})$. We have to change the solutions of the system of the linear equations.

After solving the system of linear equations for a sum rule order one of $\tilde{\mathbf{P}}$, for a sum rule order one of \mathbf{P} , and for a vanishing moment order one of \mathbf{Q} , we can find the parameters. we have we have 4 free parameters $b_{22}, d_{12}, u_{21}, u_{22}$.

$$\begin{aligned} b_{11} &= 2, \quad b_{12} = -2d_{12}, \quad b_{21} = 0, \quad d_{11} = \frac{-1}{2}, \\ d_{21} &= 0, \quad d_{22} = \frac{1}{2}, \quad u_{11} = \frac{1}{2}, \quad u_{12} = 0. \end{aligned}$$

If we choose

$$b_{22} = \frac{1081}{672}, \quad d_{12} = \frac{342}{47}, \quad u_{21} = \frac{-2}{157}, \quad u_{22} = \frac{-33}{73}.$$

by using the smoothness formula, we can obtain $\tilde{\Phi} \in W^{0.2098}$ which is supported on $[-2, 2]$ and $\Phi \in W^{0.0073}$ and it is in $L_2(\mathbb{R})$.

The lowpass analysis mask \mathbf{P} and the lowpass synthesis mask $\tilde{\mathbf{P}}$ will be as follows:

$$\mathbf{P}(\omega) = \begin{bmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} P_{11}(\omega) &= 0.25(e^{i\omega} + e^{-i\omega}) + 0.5, \\ P_{12}(\omega) &= 0, \\ P_{21}(\omega) &= -2.2617355876158e^{i\omega} + 4.5234711752317 - 2.2617355876158e^{-i\omega}, \\ P_{22}(\omega) &= -0.3108233117483(e^{i\omega} + e^{-i\omega}) + 0.6216466234967. \end{aligned}$$

$$\tilde{\mathbf{P}}(\omega) = \begin{bmatrix} \tilde{P}_{11}(\omega) & \tilde{P}_{12}(\omega) \\ \tilde{P}_{21}(\omega) & \tilde{P}_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} \tilde{P}_{11}(\omega) &= -0.125(e^{2i\omega} + e^{-2i\omega}) + 0.25(e^{i\omega} + e^{-i\omega}) + 0.75, \\ \tilde{P}_{12}(\omega) &= 1.8191489361702(e^{2i\omega} + e^{-2i\omega}) - 3.6382978723404, \\ \tilde{P}_{21}(\omega) &= 0.003184713375(e^{2i\omega} + e^{-2i\omega}) - 0.00636942675159(e^{i\omega} + e^{-i\omega}) + 0.0063694267515, \\ \tilde{P}_{22}(\omega) &= -0.15936144222683(e^{2i\omega} + e^{-2i\omega}) - 0.2260273972602(e^{i\omega} + e^{-i\omega}) + 0.4855925917368. \end{aligned}$$

Thus, the resulting Φ is supported on $[-1, 1]$. The highpass analysis mask \mathbf{Q} and the highpass synthesis mask $\tilde{\mathbf{Q}}$ will be as follows:

$$\mathbf{Q}(\omega) = \begin{bmatrix} Q_{11}(\omega) & Q_{12}(\omega) \\ Q_{21}(\omega) & Q_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} Q_{11}(\omega) &= -0.1538119183135(e^{i\omega} + e^{-3i\omega}) + 0.6923761633728e^{-i\omega} - 0.1923761633728(1 + e^{-2i\omega}), \\ Q_{12}(\omega) &= 0.0039595326337(e^{i\omega} + e^{-3i\omega}) - 0.0079190652674e^{-i\omega} + 0.0079190652674(1 + e^{-2i\omega}), \\ Q_{21}(\omega) &= -1.02242841631951(e^{i\omega} + e^{-3i\omega}) + -2.0448568326390e^{-i\omega} + 2.0448568326390(1 + e^{-2i\omega}), \\ Q_{22}(\omega) &= -0.1405091683246(e^{i\omega} + e^{-3i\omega}) + 0.7189816633507e^{-i\omega} + 0.2810183366492(1 + e^{-2i\omega}). \end{aligned}$$

and

$$\tilde{\mathbf{Q}}(\omega) = \begin{bmatrix} 0.5e^{-i\omega} - 0.25(1 + e^{-2i\omega}) & 3.6382978723404(1 + e^{-2i\omega}) \\ 0 & 0.5e^{-i\omega} + 0.25(1 + e^{-2i\omega}) \end{bmatrix}.$$

Next, we will consider a 3-step multiwavelet multiresolution algorithm and obtain the corresponding biorthogonal multifilter banks to this algorithm so we can obtain framelets with a higher smoothness order.

3.2.2 3-Step Multiwavelet Algorithm

The decomposition algorithm of this 3-step algorithm is given in (3.31)-(3.33) as shown in Fig. 3.6. In Step 1 given by formula (3.31), we replace each \mathbf{v} associated with an even node $2k$ by \mathbf{v}'' . Then, with the obtained \mathbf{v}'' , we obtain $\tilde{\mathbf{e}}$ that is associated with odd nodes $2k + 1$ by (3.32). Finally, with the obtained \mathbf{v}'' in Step 1, we replace it by $\tilde{\mathbf{v}}$ in (3.33).

3-step Multiwavelet Decomposition Algorithm:

$$\text{Step 1. } \mathbf{v}'' = \{\mathbf{v} - (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{D}\} \mathbf{B}^{-1}; \quad (3.31)$$

$$\text{Step 2. } \tilde{\mathbf{e}} = \mathbf{e} - (\mathbf{v}_0'' + \mathbf{v}_1'') \mathbf{U}; \quad (3.32)$$

$$\text{Step 3. } \tilde{\mathbf{v}} = \mathbf{v}'' - (\tilde{\mathbf{e}}_{-1} + \tilde{\mathbf{e}}_0) \mathbf{D}_1 - (\tilde{\mathbf{e}}_{-2} + \tilde{\mathbf{e}}_1) \mathbf{C}_1. \quad (3.33)$$

3-step Multiwavelet Reconstruction Algorithm:

$$\text{Step 1. } \mathbf{v}'' = \tilde{\mathbf{v}} + (\tilde{\mathbf{e}}_{-1} + \tilde{\mathbf{e}}_0) \mathbf{D}_1 + (\tilde{\mathbf{e}}_{-2} + \tilde{\mathbf{e}}_1) \mathbf{C}_1; \quad (3.34)$$

$$\text{Step 2. } \mathbf{e} = \tilde{\mathbf{e}} + (\mathbf{v}_0'' + \mathbf{v}_1'') \mathbf{U}; \quad (3.35)$$

$$\text{Step 3. } \mathbf{v} = \mathbf{v}'' \mathbf{B} + (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{D}, \quad (3.36)$$

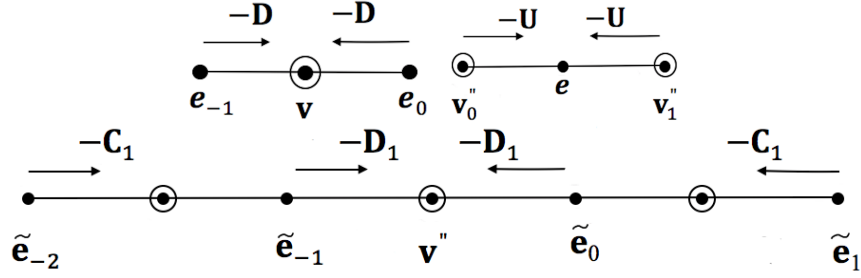


Figure 3.6: Top left: Decomposition Step 1; top right: Decomposition Step 2; bottom: Decomposition Step 3.

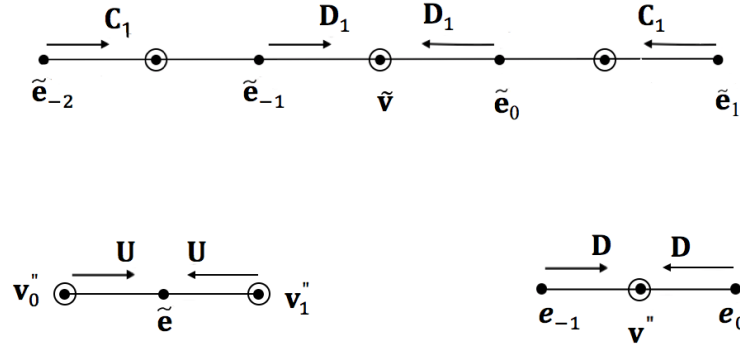


Figure 3.7: Top: Reconstruction Step 1; bottom-left: Reconstruction Step 2; bottom-right: Reconstruction Step 3.

where \mathbf{B} , \mathbf{D} , \mathbf{U} , \mathbf{D}_1 and \mathbf{C}_1 are 2×2 matrices and the entries of these matrices are some constants in \mathbb{R} .

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} d_{1,11} & d_{1,12} \\ d_{1,21} & d_{1,22} \end{bmatrix},$$

$$\mathbf{C}_1 = \begin{bmatrix} c_{1,11} & c_{1,12} \\ c_{1,21} & c_{1,22} \end{bmatrix}.$$

Fig. 3.7 shows that the backward algorithm of the decomposition algorithm is the reconstruction algorithm. The multiresolution reconstruction algorithm is given in (3.34)-(3.36), where the matrices are the same 2×2 matrices in the decomposition algorithm.

First in Step 1, we update $\tilde{\mathbf{v}}$ of the lowpass output by \mathbf{v}'' given in (3.34). Then, in Step 2 with the obtained \mathbf{v}'' in Step 1, we obtain \mathbf{e} given by formula (3.35). Finally, we replace \mathbf{v}'' , which is obtained in Step 1 by \mathbf{v} given in (3.36). So the inputs here are $\tilde{\mathbf{v}}$, $\tilde{\mathbf{e}}$ and the outputs are \mathbf{v}'' , \mathbf{e} , \mathbf{v} .

After following the same calculation details in 2-step frame algorithm, we can obtain the analysis and synthesis multifilter banks. Denote

$$\begin{cases} \mathbf{A}_2(2\omega) = \begin{bmatrix} \mathbf{I}_2 & -(1 + e^{2i\omega})\mathbf{D}_1^T - (e^{-2i\omega} + e^{4i\omega})\mathbf{C}_1^T \\ \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{4 \times 4}, \\ \mathbf{A}_1(2\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 \\ -(1 + e^{-2i\omega})\mathbf{U}^T & \mathbf{I}_2 \end{bmatrix}_{4 \times 4}, \\ \mathbf{A}_0(2\omega) = \begin{bmatrix} (\mathbf{B}^{-1})^T & -(1 + e^{2i\omega})(\mathbf{B}^{-1})^T\mathbf{D}^T \\ \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{4 \times 4}, \end{cases} \quad (3.37)$$

$$\begin{cases} \tilde{\mathbf{A}}_2(2\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 \\ (1 + e^{-2i\omega})\mathbf{D}_1 + (e^{2i\omega} + e^{-4i\omega})\mathbf{C}_1 & \mathbf{I}_2 \end{bmatrix}_{4 \times 4}, \\ \tilde{\mathbf{A}}_1(2\omega) = \begin{bmatrix} \mathbf{I}_2 & (1 + e^{2i\omega})\mathbf{U} \\ \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{4 \times 4}, \\ \tilde{\mathbf{A}}_0(2\omega) = \begin{bmatrix} \mathbf{B} & \mathbf{0}_2 \\ (1 + e^{-2i\omega})\mathbf{D} & \mathbf{I}_2 \end{bmatrix}_{4 \times 4}. \end{cases} \quad (3.38)$$

Then $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}\}$ can be written as

$$\begin{bmatrix} \mathbf{P}(\omega) \\ \mathbf{Q}(\omega) \end{bmatrix}_{4 \times 2} = \mathbf{A}_2(2\omega)\mathbf{A}_1(2\omega)\mathbf{A}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}_{4 \times 2}, \quad (3.39)$$

and

$$\begin{bmatrix} \tilde{\mathbf{P}}(\omega) \\ \tilde{\mathbf{Q}}(\omega) \end{bmatrix}_{4 \times 2} = \frac{1}{2}\tilde{\mathbf{A}}_2(2\omega)\tilde{\mathbf{A}}_1(2\omega)\tilde{\mathbf{A}}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}_{4 \times 2}. \quad (3.40)$$

After solving the system of linear equations for a sum rule order three of $\tilde{\mathbf{P}}$, for a sum rule order two of \mathbf{P} , and for a vanishing moment order one of \mathbf{Q} , we can find the parameters. we have 9 free parameters which are c_{12} , c_{22} , $d_{1,11}$, $d_{1,12}$, $d_{1,21}$, $d_{1,22}$, d_{11} , d_{22} , u_{21} , and the

other parameters are

$$\begin{aligned}
a &= 0, \quad b_{11} = \frac{1}{4}(512d_{11}^2d_{22}u_{21} - 192d_{11}^2d_{22} - 128d_{11}^2u_{21} - 320d_{11}d_{22}u_{21} + 40d_{11}^2 + 120d_{11}d_{22} \\
&\quad + 192d_{11}u_{21} + 48d_{22}u_{21} - 42d_{11} - 18d_{22} - 56u_{21} + 11)/(4d_{11} - 1), \quad b_{12} = (\frac{1}{2}(32d_{11}d_{22}u_{21} \\
&\quad - 12d_{11}d_{22} - 16d_{11}u_{21} - 8d_{22}u_{21} + 6d_{11} + 3d_{22} + 12u_{21} - 3))/(4d_{11} - 1), \\
b_{21} &= -(\frac{1}{4}(512d_{11}^2d_{22}u_{21} + 64d_{11}^2d_{22} - 128d_{11}^2u_{21} - 320d_{11}d_{22}u_{21} - 24d_{11}^2 - 40d_{11}d_{22} + 192d_{11}u_{21} \\
&\quad + 48d_{22}u_{21} - 10d_{11} + 6d_{22} - 56u_{21} + 7))/(4d_{11} - 1), \\
b_{22} &= -(\frac{1}{2}(32d_{11}d_{22}u_{21} + 4d_{11}d_{22} - 16d_{11}u_{21} - 8d_{22}u_{21} - 2d_{11} - d_{22} + 12u_{21} - 1))/(4d_{11} - 1), \\
c &= 0, \quad c_{11} = -c_{12} - d_{1,11} - d_{1,12} - \frac{1}{2}d_{11} - \frac{1}{4}, \quad c_{21} = -4d_{11}d_{22} - c_{22} - d_{1,21} - d_{1,22} + \frac{3}{2}d_{11} + d_{22} - \frac{3}{4}, \\
d_{12} &= 0, \quad d_{21} = 8d_{11}d_{22} - 3d_{11} - 3d_{22} + 1, \\
g &= -(\frac{2}{3}(64d_{11}d_{22}u_{21} - 24d_{11}d_{22} - 16d_{11}u_{21} - 16d_{22}u_{21} + 6d_{11} + 6d_{22} + 12u_{21} - 3))/(16u_{21} \\
&\quad - 3), \quad h = -(\frac{1}{3}(128d_{11}d_{22}u_{21} - 32d_{11}u_{21} - 32d_{22}u_{21} + 24u_{21} - 3))/(16u_{21} - 3), \\
u_{11} &= -u_{21} + \frac{1}{2}, \quad u_{12} = -u_{21} + \frac{3}{8}, \quad u_{22} = u_{21} + \frac{1}{8}.
\end{aligned}$$

If we choose

$$\begin{aligned}
[c_{12}, c_{22}, d_{1,11}, d_{1,12}, d_{1,21}, d_{1,22}, d_{11}, d_{22}, u_{21}] &= \left[\frac{463}{3705}, \frac{-137}{889}, \frac{51}{148}, \frac{-752}{1613}, \frac{-605}{711}, \frac{1359}{2039}, \frac{416}{679} \right. \\
&\quad , \\
&\quad \left. \frac{-33}{247}, \frac{-306}{1399} \right].
\end{aligned}$$

and by using the smoothness formula, we can obtain $\tilde{\Phi} \in W^{1.7107}$ which is supported on $[-2, 2]$ and $\Phi \in W^{0.3283}$ and it is in $L_2(\mathbb{R})$.

The lowpass analysis mask \mathbf{P} and the lowpass synthesis mask $\tilde{\mathbf{P}}$ will be as follows:

$$\mathbf{P}(\omega) = \begin{bmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned}
P_{11}(\omega) &= 0.009913974865(e^{5i\omega} + e^{-5i\omega}) + 0.01618170417(e^{4i\omega} + e^{-4i\omega}) - 0.2963693161(e^{3i\omega} + e^{-3i\omega}) \\
&\quad - 0.4135997606(e^{2i\omega} + e^{-2i\omega}) - 0.1465027480(e^{i\omega} + e^{-i\omega}) - 0.7369275583, \\
P_{12}(\omega) &= 0.02179608381(e^{5i\omega} + e^{-5i\omega}) - 0.03087750317(e^{4i\omega} + e^{-4i\omega}) - 0.3325092568(e^{3i\omega} + e^{-3i\omega}) \\
&\quad - 0.7831857424(e^{2i\omega} + e^{-2i\omega}) - 0.5974292849(e^{i\omega} + e^{-i\omega}) + 1.528643697, \\
P_{21}(\omega) &= 0.003695006759(e^{5i\omega} + e^{-5i\omega}) - 0.006031032666(e^{4i\omega} + e^{-4i\omega}) + 0.003695006759(e^{3i\omega} + e^{-3i\omega}) \\
&\quad - 0.07254899017(e^{2i\omega} + e^{-2i\omega}) + 0.1831753783(e^{i\omega} + e^{-i\omega}) - 0.5345246745 \\
P_{22}(\omega) &= -0.006876652381(e^{5i\omega} + e^{-5i\omega}) + 0.002175413577(e^{4i\omega} + e^{-4i\omega}) - 0.09200852751(e^{3i\omega} + e^{-3i\omega}) \\
&\quad + 0.1796693933(e^{2i\omega} + e^{-2i\omega}) - 0.1218479871(e^{i\omega} + e^{-i\omega}) + 0.1209848222.
\end{aligned}$$

and

$$\tilde{\mathbf{P}}(\omega) = \begin{bmatrix} \tilde{P}_{11}(\omega) & \tilde{P}_{12}(\omega) \\ \tilde{P}_{21}(\omega) & \tilde{P}_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} \tilde{P}_{11}(\omega) &= -0.01040114408(e^{2i\omega} + e^{-2i\omega}) + 0.3593638313(e^{i\omega} + e^{-i\omega}) + 0.1381020689, \\ \tilde{P}_{12}(\omega) &= -0.03966196936(e^{2i\omega} + e^{-2i\omega}) + 0.2968638313(e^{i\omega} + e^{-i\omega}) + 0.1129935650, \\ \tilde{P}_{21}(\omega) &= -0.01582721164(e^{2i\omega} + e^{-2i\omega}) - 0.1093638313(e^{i\omega} + e^{-i\omega}) + 0.8777793748, \\ \tilde{P}_{22}(\omega) &= 0.006261159648(e^{2i\omega} + e^{-2i\omega}) - 0.04686383131(e^{i\omega} + e^{-i\omega}) + 0.04686383131. \end{aligned}$$

The highpass analysis mask \mathbf{Q} and the highpass synthesis mask $\tilde{\mathbf{Q}}$ will be as follows:

$$\mathbf{Q}(\omega) = \begin{bmatrix} Q_{11}(\omega) & Q_{12}(\omega) \\ Q_{21}(\omega) & Q_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} Q_{11}(\omega) &= -0.7639542186(e^{i\omega} + e^{-3i\omega}) - 0.5279084372e^{-i\omega} + 1.24693489(1 + e^{-2i\omega}), \\ Q_{12}(\omega) &= 1.051458929(e^{i\omega} + e^{-3i\omega}) + 2.102917857e^{-i\omega} - 2.321944310(1 + e^{-2i\omega}), \\ Q_{21}(\omega) &= -0.5955229056(e^{i\omega} + e^{-3i\omega}) - 1.191045811e^{-i\omega} + 0.9720193579(1 + e^{-2i\omega}), \\ Q_{22}(\omega) &= 0.8080181956(e^{i\omega} + e^{-3i\omega}) - 2.616036391e^{-i\omega} - 1.897009938(1 + e^{-2i\omega}), \end{aligned}$$

and

$$\tilde{\mathbf{Q}}(\omega) = \begin{bmatrix} \tilde{Q}_{11}(\omega) & \tilde{Q}_{12}(\omega) \\ \tilde{Q}_{21}(\omega) & \tilde{Q}_{22}(\omega) \end{bmatrix}.$$

$$\begin{aligned} \tilde{Q}_{11}(\omega) &= 0.1585230599(e^{2i\omega} + e^{-4i\omega}) + 0.05623542985(e^{4i\omega} + e^{-6i\omega}) \\ &\quad - 0.1227152000, \\ \tilde{Q}_{12}(\omega) &= -0.02311605936(e^{2i\omega} + e^{-4i\omega}) + 0.02298051103(e^{4i\omega} + e^{-6i\omega}) \\ &\quad - 0.1662396185, \\ \tilde{Q}_{21}(\omega) &= -0.1541698312(e^{2i\omega} + e^{-4i\omega}) - 0.07050712629(e^{4i\omega} + e^{-6i\omega}) \\ &\quad - 0.1639944150, \\ \tilde{Q}_{22}(\omega) &= 0.04723320133(e^{2i\omega} + e^{-4i\omega}) - 0.02878095098(e^{4i\omega} + e^{-6i\omega}) \\ &\quad + 0.1486561310. \end{aligned}$$

When we use the obtained parameters to construct biorthogonal multiwavelets, we should

select the parameters such that the multiscaling function $\tilde{\Phi}$ (synthesis) is smoother than the analysis multiscaling function Φ . Also, the analysis highpass multifilters should have higher vanishing moments. That means $\tilde{\mathbf{P}}$ has a higher sum rule order than \mathbf{P} . Thus, \mathbf{Q} has higher vanishing moment order than $\tilde{\mathbf{Q}}$. By minimization (using Matlab), we can select the remaining parameters such that the multiscaling function Φ and/or its dual $\tilde{\Phi}$ have optimal smoothness.

When we want to obtain biorthogonal multiwavelets with a higher smoothness order and/or higher vanishing moment orders, we need to use algorithms with more iterative steps.

Chapter 4

One-dimensional Multiple Bi-frames with Uniform Symmetry for Curve Multiresolution Processing

4.1 Introduction

In [73], Jiang has presented the construction of biorthogonal wavelet and bi-frames and each framelet is symmetric in the scalar case and the wavelet bi-frames are univariate. By following the work in [73], we present the construction of biorthogonal wavelet and bi-frames in multiwavelet case (vector case).

We develop one-dimensional multiple bi-frames and associated multiresolution algorithm templates. In Subsection 4.2, we discuss and obtain some results of multiple bi-frames with uniform symmetry: type I. We will discuss 2-step type I multiple bi-frame algorithm, 3-step type I multiple bi-frame algorithm, and uniform symmetry. In addition, we discuss and obtain some results of multiple bi-frames with uniform symmetry: type II. We will discuss 2-step type II multiple bi-frame algorithm, 3-step type II multiple bi-frame algorithm, and uniform symmetry.

We will introduce bi-frames with 2 framelets $\{\Psi^{(1)}, \Psi^{(2)}\}$ (frame generators). We consider one-dimensional (1-D for short) bi-frame multiresolution algorithms in the vector case. First, we associate the lowpass outputs and highpass outputs to the nodes of \mathbf{Z} and then we can represent multiple bi-frame multiresolution algorithms by symmetric templates by using the method of the lifting scheme. From the algorithms templates we can derive the corresponding multifilter banks and then design multiple bi-frames based on the multifilter banks' smoothness and vanishing moments.

From [71], $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ is called a **multiwavelet filter bank** (or **multifilter bank**), and \mathbf{P} is called a **matrix lowpass filter** and $\mathbf{Q}^{(\ell)}, \ell = 1, 2$ **matrix highpass filters**. The matrix filters $\mathbf{P}, \mathbf{Q}^{(\ell)}$ are called **finite impulse response (FIR) filters** if there exists an integer N such that $\mathbf{P}_k = \mathbf{0}, \mathbf{Q}_k^{(\ell)} = \mathbf{0}, |k| > N$.

Assume $\mathbf{P}(\omega)$ is 2×2 matrix symbol, which is the finite impulse response (FIR) filter with its impulse response 2×2 matrix coefficients \mathbf{P}_k , for a matrix sequence $\{\mathbf{P}_k\}_{k \in \mathbf{Z}}$ on \mathbb{R} with finite nonzero \mathbf{P}_k , (a factor $1/2$ is multiplied for convenience):

$$\mathbf{P}(\omega) = \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{P}_k e^{-ik\omega}, \quad \omega \in \mathbb{R}. \quad (4.1)$$

Assume $\mathbf{Q}_k^{(\ell)}$ are 2×2 matrix coefficients and $\ell = 1, 2$.

$$\mathbf{Q}^{(\ell)}(\omega) = \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{Q}_k^{(\ell)} e^{-ik\omega}, \quad \omega \in \mathbb{R}, \quad \ell = 1, 2. \quad (4.2)$$

The modulation matrices play an important role in formulating the wavelet decomposition and reconstruction algorithms. In [3] introduces the modulation matrices in the scalar case. Now, we provide the modulation matrices in the multiwavelet case (or vector case). Let $\mathbf{M}_{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}}$ and $\mathbf{M}_{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}}$ be the modulation matrices of FIR multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$, respectively, defined by

$$\mathbf{M}_{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}} := \begin{bmatrix} \mathbf{P}(\omega) & \mathbf{P}(\omega + \pi) \\ \mathbf{Q}^{(1)}(\omega) & \mathbf{Q}^{(1)}(\omega + \pi) \\ \mathbf{Q}^{(2)}(\omega) & \mathbf{Q}^{(2)}(\omega + \pi) \end{bmatrix}_{6 \times 4}, \quad (4.3)$$

$$\mathbf{M}_{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}} := \begin{bmatrix} \tilde{\mathbf{P}}(\omega) & \tilde{\mathbf{P}}(\omega + \pi) \\ \tilde{\mathbf{Q}}^{(1)}(\omega) & \tilde{\mathbf{Q}}^{(1)}(\omega + \pi) \\ \tilde{\mathbf{Q}}^{(2)}(\omega) & \tilde{\mathbf{Q}}^{(2)}(\omega + \pi) \end{bmatrix}_{6 \times 4}. \quad (4.4)$$

In the multiwavelet setting, a pair of frame multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}, \{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ is said to be biorthogonal if $\mathbf{M}_{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}}$ and $\mathbf{M}_{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}}$ defined by (4.3) and (4.4), respectively, satisfy

$$\mathbf{M}_{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}}(\omega)^* \mathbf{M}_{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}}(\omega) = \mathbf{I}_4, \quad \omega \in \mathbb{R}, \quad (4.5)$$

\mathbf{M}^* denotes the complex conjugate and transpose of a matrix \mathbf{M} .

Therefore, a pair of FIR multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ is said to be **biorthogonal** frame multifilter bank if it satisfies the **biorthogonality conditions**

$$\begin{cases} \tilde{\mathbf{P}}(\omega)^* \mathbf{P}(\omega) + \sum_{\ell=1}^2 \tilde{\mathbf{Q}}^{(\ell)}(\omega)^* \mathbf{Q}^{(\ell)}(\omega) = \mathbf{I}_2, \\ \tilde{\mathbf{P}}(\omega)^* \mathbf{P}(\omega + \pi) + \sum_{\ell=1}^2 \tilde{\mathbf{Q}}^{(\ell)}(\omega)^* \mathbf{Q}^{(\ell)}(\omega + \pi) = \mathbf{0}_2. \end{cases} \quad (4.6)$$

Suppose $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ is a pair of biorthogonal frame multifilter banks (also called FIR multifilter banks), Let Φ and $\tilde{\Phi}$, $\Phi := [\phi_1, \phi_2]^T$, $\tilde{\Phi} := [\tilde{\phi}_1, \tilde{\phi}_2]^T$,

denote the associated refinable function vectors satisfying the refinement equations

$$\Phi(x) = \sum_{k \in \mathbf{Z}} \mathbf{P}_k \Phi(2x - k), \quad \tilde{\Phi}(x) = \sum_{k \in \mathbf{Z}} \tilde{\mathbf{P}}_k \tilde{\Phi}(2x - k), \quad x \in \mathbb{R}. \quad (4.7)$$

We focus on frames with two wavelet framelets $\{\Psi^{(1)}, \Psi^{(2)}\}$, which are function vectors. Let $\Psi^{(\ell)}, \tilde{\Psi}^{(\ell)}, \ell = 1, 2$ be function vectors defined by

$$\Psi^{(\ell)}(x) = \sum_{k \in \mathbf{Z}} \mathbf{Q}_k^{(\ell)} \Phi(2x - k), \quad \tilde{\Psi}^{(\ell)}(x) = \sum_{k \in \mathbf{Z}} \tilde{\mathbf{Q}}_k^{(\ell)} \tilde{\Phi}(2x - k), \quad x \in \mathbb{R}. \quad (4.8)$$

We say that $\Psi_i^{(\ell)}, \tilde{\Psi}_i^{(\ell)}, \ell = 1, 2, i = 1, 2$ generate multiple bi-frames (bi-frames) of $L_2(\mathbb{R})$ or dual multiple frames of $L_2(\mathbb{R})$ if

$\{\psi_{i,j,k}^{(1)}(x), \psi_{i,j,k}^{(2)}(x)\}_{i=1,2, j,k \in \mathbf{Z}}$ and $\{\tilde{\psi}_{i,j,k}^{(1)}(x), \tilde{\psi}_{i,j,k}^{(2)}(x)\}_{i=1,2, j,k \in \mathbf{Z}}$ are frames of $L_2(\mathbb{R})$ and that for any function $f \in L_2(\mathbb{R})$, f can be written as (in L^2 -norm)

$$f = \sum_{\ell=1}^2 \sum_{i=1}^2 \sum_{j,k \in \mathbf{Z}} \langle f, \tilde{\psi}_{i,j,k}^{(\ell)} \rangle \psi_{i,j,k}^{(\ell)}. \quad (4.9)$$

When we use a frame multifilter bank $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ as the analysis multifilter bank, the multiple frame multiresolution decomposition algorithm for input data $\{\mathbf{c}_k\}$ is

$$\tilde{\mathbf{c}}_n = \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{c}_k \mathbf{P}_{k-2n}^T, \quad \mathbf{d}_n^{(1)} = \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{c}_k \mathbf{Q}_{k-2n}^{(1)T}, \quad \mathbf{d}_n^{(2)} = \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{c}_k \mathbf{Q}_{k-2n}^{(2)T}, \quad (4.10)$$

where $\tilde{\mathbf{c}}_n, \mathbf{c}_k, \mathbf{d}_n^{(1)}$ and $\mathbf{d}_n^{(2)}$ are 1×2 “row-vectors” and the masks $\mathbf{P}_{k-2n}, \mathbf{Q}_{k-2n}^{(1)}$ and $\mathbf{Q}_{k-2n}^{(2)}$ are 2×2 matrices.

If an FIR frame multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ is biorthogonal to $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$, then the input data $\{\mathbf{c}_k\}$ can be recovered from $\tilde{\mathbf{c}}_n$ and $\mathbf{d}_n^{(1)}, \mathbf{d}_n^{(2)}$ by the multiple frame multiresolution reconstruction algorithm:

$$\mathbf{c}'_k = \sum_{n \in \mathbf{Z}} \tilde{\mathbf{c}}_n \tilde{\mathbf{P}}_{k-2n} + \sum_{n \in \mathbf{Z}} \mathbf{d}_n^{(1)} \tilde{\mathbf{Q}}_{k-2n}^{(1)} + \sum_{n \in \mathbf{Z}} \mathbf{d}_n^{(2)} \tilde{\mathbf{Q}}_{k-2n}^{(2)}, \quad k \in \mathbf{Z}, \quad (4.11)$$

where a frame multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ is called the (frame) synthesis multifilter bank. $\{\tilde{\mathbf{c}}_k\}_k$ is called the lowpass output and $\{\mathbf{d}_k^{(1)}\}_k, \{\mathbf{d}_k^{(2)}\}_k$ are called the highpass outputs of $\{\mathbf{c}_k\}_k$. $\{\tilde{\mathbf{c}}_k\}_k$ is called the “approximation” of $\{\mathbf{c}_k\}_k$, $\{\mathbf{d}_k^{(1)}\}_k$ and $\{\mathbf{d}_k^{(2)}\}_k$ are the “details” of $\{\mathbf{c}_k\}_k$.

Theorem 2. *If $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ are biorthogonal frame multifilter bank, then the input data $\{\mathbf{c}_k\}_k$ can be recovered from its approximation $\{\tilde{\mathbf{c}}_k\}_k$ and details $\{\mathbf{d}_k^{(1)}\}_k$ and $\{\mathbf{d}_k^{(2)}\}_k$, namely: the input data $\{\mathbf{c}_k\}_k$ is exactly $\{\mathbf{c}'_k\}_k$.*

Proof: Let $\tilde{\mathbf{c}}(2\omega)$, $\mathbf{d}^{(1)}(2\omega)$, $\mathbf{d}^{(2)}(2\omega)$, $\mathbf{c}(\omega)$, and $\mathbf{c}'(\omega)$ denote the Z-transforms of $\tilde{\mathbf{c}}_n$, $\mathbf{d}_n^{(1)}$, $\mathbf{d}_n^{(2)}$, \mathbf{c}_k , and \mathbf{c}'_k respectively. The Z-transforms $\mathbf{c}(\omega)$ of $\mathbf{c}_k = \{\mathbf{c}_k\}_k$ is $\mathbf{c}(\omega) = \frac{1}{2} \sum_{k \in \mathbf{Z}} \mathbf{c}_k e^{-ik\omega}$. In the frequency domain, we have

$$\tilde{\mathbf{c}}(2\omega) = \frac{1}{2} \left(\mathbf{c}(\omega) \overline{\mathbf{P}(\omega)}^T + \mathbf{c}(\omega + \pi) \overline{\mathbf{P}(\omega + \pi)}^T \right),$$

$$\mathbf{d}^{(1)}(2\omega) = \frac{1}{2} \left(\mathbf{c}(\omega) \overline{\mathbf{Q}^{(1)}(\omega)}^T + \mathbf{c}(\omega + \pi) \overline{\mathbf{Q}^{(1)}(\omega + \pi)}^T \right),$$

$$\mathbf{d}^{(2)}(2\omega) = \frac{1}{2} \left(\mathbf{c}(\omega) \overline{\mathbf{Q}^{(2)}(\omega)}^T + \mathbf{c}(\omega + \pi) \overline{\mathbf{Q}^{(2)}(\omega + \pi)}^T \right),$$

and

$$\mathbf{c}'(\omega) = 2 \left(\tilde{\mathbf{c}}(2\omega) \tilde{\mathbf{P}}(\omega) + \mathbf{d}^{(1)}(2\omega) \tilde{\mathbf{Q}}^{(1)}(\omega) + \mathbf{d}^{(2)}(2\omega) \tilde{\mathbf{Q}}^{(2)}(\omega) \right).$$

Suppose that the frame multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ are biorthogonal. By plugging $\tilde{\mathbf{c}}(2\omega)$, $\mathbf{d}^{(1)}(2\omega)$, and $\mathbf{d}^{(2)}(2\omega)$ into $\mathbf{c}'(\omega)$ and if the conditions in (4.6) hold, then $\mathbf{c}'(\omega) = \mathbf{c}(\omega)$. Hence, the biorthogonality of $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ implies that the original data can be recovered from the lowpass and highpass outputs by the multiple frame reconstruction algorithms.

We use multifilter banks for curve multiresolution processing. There are two important points we should consider. The first one is the easy implementation of algorithms which can be done if the algorithms given by templates. The second point is the symmetry of the filters. So, it is required that all the 1-D algorithms templates of the analysis algorithms and synthesis algorithms to be symmetric. That means the multiscaling functions Φ , $\tilde{\Phi}$ and all the multiwavelets $\Psi^{(\ell)}$, $\tilde{\Psi}^{(\ell)}$, $\ell = 1, 2$ are symmetric. Thus, we say that a multiple bi-frame has uniform symmetry if its associated multiscaling refinable function and each of its framelets are symmetric too.

4.2 Multiple Bi-frames with Uniform Symmetry: Type I

A multiresolution algorithm can be given by templates to make it easy for implementation. Also we can represent the multiresolution algorithm by some templates and the frame algorithms given by many iterative steps. Every step in the algorithms will be represented by a symmetric template.

Suppose $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ are a pair of biorthogonal frame multifilter banks. Let $\tilde{\mathbf{c}}_k$ be the lowpass output and $\mathbf{d}_k^{(1)}$, $\mathbf{d}_k^{(2)}$ be the highpass outputs of the input \mathbf{c}_k defined by (4.10). For initial data $\{\mathbf{c}_k\}$, let

$$\mathbf{v}_k = \mathbf{c}_{2k}, \mathbf{e}_k = \mathbf{c}_{2k+1}, k \in \mathbf{Z}, \quad (4.12)$$

and denote

$$\tilde{\mathbf{v}}_k = \tilde{\mathbf{c}}_k, \tilde{\mathbf{f}}_k = \mathbf{d}_k^{(1)}, \tilde{\mathbf{e}}_k = \mathbf{d}_k^{(2)}, \quad k \in \mathbf{Z}, \quad (4.13)$$

where \mathbf{v}_k , \mathbf{e}_k , $\tilde{\mathbf{v}}_k$, $\tilde{\mathbf{f}}_k$ and $\tilde{\mathbf{e}}_k$ are 1×2 row-vectors

Associating $\tilde{\mathbf{c}}_k$ (lowpass output) and $\mathbf{d}_k^{(1)}$, $\mathbf{d}_k^{(2)}$ (the highpass outputs) of the input \mathbf{c}_k defined by (4.10) with the analysis multiple frame multifilter bank $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ with the nodes of \mathbf{Z} , we can show the multiple frame multiresolution algorithm templates. Other ways to associate $\tilde{\mathbf{v}}_k$ and $\tilde{\mathbf{f}}_k$, $\tilde{\mathbf{e}}_k$ with the nodes of \mathbf{Z} will give us different templates for the decomposition algorithm and the reconstruction algorithm. In this research, we have two kind of the association that give us two types of framelets, called type I and type II framelets. In type I, we associate $\tilde{\mathbf{v}}_k (= \tilde{\mathbf{c}}_k)$ with an even node $2k$ and both $\tilde{\mathbf{f}}_k$ (one highpass output $\mathbf{d}_k^{(1)}$) and $\tilde{\mathbf{e}}_k$ (the other highpass output $\mathbf{d}_k^{(2)}$) with an odd node $2k + 1$. Then, the decomposition and reconstruction algorithm are represented by templates for this type. In type II, we associate both $\tilde{\mathbf{v}}_k (= \tilde{\mathbf{c}}_k)$ and $\tilde{\mathbf{f}}_k$ (one highpass output $\mathbf{d}_k^{(1)}$) with an even node $2k$ and $\tilde{\mathbf{e}}_k$ (the other highpass output $\mathbf{d}_k^{(2)}$) with an odd node $2k + 1$. Thus, the decomposition algorithm and reconstruction algorithms can be represented by templates for this type.

Construction uniformly symmetric multiple bi-frames of type I and II is similar to construction biorthogonal multiwavelets. The first step, begin with symmetric templates of small sizes of the decomposition and reconstruction algorithms. The algorithm templates are given by many iterative steps as symmetric templates. The second step, we can get the corresponding multiple bi-frame multifilter banks that are given by some obtained parameters. Finally, the selection of the suitable parameters depends on the smoothness and vanishing moments of multiple frames.

In this section, we will investigate multiple bi-frames of type I and the next section multiple bi-frames of type II.

4.2.1 2-Step Type I Multiple Bi-frame Algorithm

In this subsection, let us consider a 2-step type I multiple frame multiresolution algorithm. The decomposition algorithm of this 2-step algorithm (4.14)-(4.15) as shown in Fig. 4.1. The decomposition Step1 given in (4.14), we obtain lowpass output $\tilde{\mathbf{v}}$ by replacing each \mathbf{v} associated with an even node $2k$ by $\tilde{\mathbf{v}}$. Then, in Step 2, we use the obtained $\tilde{\mathbf{v}}$ to obtain two highpass outputs $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{e}}$ associated with odd nodes $2k + 1$ given in (4.15).

2-step Type I Multiple Bi-frame Decomposition Algorithm:

$$\text{Step 1. } \tilde{\mathbf{v}} = \frac{1}{b} \{\mathbf{v} - (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{D}\}; \quad (4.14)$$

$$\text{Step 2. } \begin{cases} \tilde{\mathbf{f}} = \mathbf{e} - (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1) \mathbf{U}, \\ \tilde{\mathbf{e}} = \mathbf{e} - (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1) \mathbf{W}. \end{cases} \quad (4.15)$$

2-step Type I Multiple Bi-frame Reconstruction Algorithm:

$$\text{Step 1. } \mathbf{e} = t\{\tilde{\mathbf{f}} + (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1)\mathbf{U}\} + (1-t)\{\tilde{\mathbf{e}} + (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1)\mathbf{W}\}; \quad (4.16)$$

$$\text{Step 2. } \mathbf{v} = b\tilde{\mathbf{v}} + (\mathbf{e}_{-1} + \mathbf{e}_0)\mathbf{D}, \quad (4.17)$$

where b, t in \mathbb{R} are some constants and

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \text{ and } \mathbf{W} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix},$$

are 2×2 matrices where the entries of these square matrices are some constants in \mathbb{R} .

The reconstruction algorithm is the backward algorithm of the decomposition algorithm as shown in Fig. 4.2. The multiresolution reconstruction algorithm is given in (4.16) and (4.17), where the matrices are the same 2×2 matrices in the decomposition algorithm.

In Step 1, we replace $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{e}}$ of the highpass outputs by \mathbf{e} , the original data \mathbf{c}_{2k+1} associated with odd nodes, see (4.16). Then in Step 2, with the obtained \mathbf{e} in Step 1, we update $\tilde{\mathbf{v}}$ of the lowpass output by \mathbf{v} by formula given in (4.17). In this step we recover original data \mathbf{c}_{2k} associated with even nodes. So the inputs here are $\tilde{\mathbf{v}}, \tilde{\mathbf{f}}, \tilde{\mathbf{e}}$ and the outputs are \mathbf{e}, \mathbf{v} .

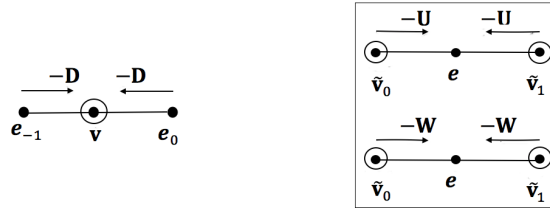


Figure 4.1: Left: Decomposition Step 1; Right: Decomposition Step 2.

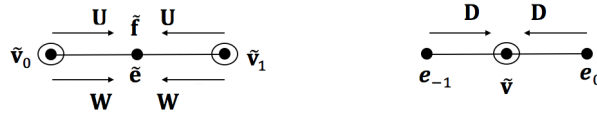


Figure 4.2: Left: Reconstruction Step 1; Right: Reconstruction Step 2.

Now, let us obtain the corresponding biorthogonal multifilter banks (in the vector case) $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ to this 2-step frame algorithm. Then, we can discuss the properties of the corresponding multiwavelet to choose the parameters.

From (4.13) and (4.14), we have

$$\tilde{\mathbf{c}}_n = \tilde{\mathbf{v}}_n = \frac{1}{b} \{ \mathbf{v}_n - (\mathbf{e}_{n-1} + \mathbf{e}_n) \mathbf{D} \} = \frac{1}{b} \{ \mathbf{c}_{2n} - (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+1}) \mathbf{D} \}. \quad (4.18)$$

From (4.13) and (4.15) we have

$$\begin{aligned} \mathbf{d}_n^{(1)} &= \tilde{\mathbf{f}}_n = \mathbf{e}_n - (\tilde{\mathbf{v}}_n + \tilde{\mathbf{v}}_{n+1}) \mathbf{U} \\ &= \mathbf{c}_{2n+1} - \left(\frac{1}{b} \{ \mathbf{c}_{2n} - (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+1}) \mathbf{D} \} + \frac{1}{b} \{ \mathbf{c}_{2n+2} - (\mathbf{c}_{2n+1} + \mathbf{c}_{2n+3}) \mathbf{D} \} \right) \mathbf{U} \\ &= \mathbf{c}_{2n+1} \left(\mathbf{I}_2 + \frac{2}{b} \mathbf{D} \mathbf{U} \right) - \frac{1}{b} (\mathbf{c}_{2n} + \mathbf{c}_{2n+2}) \mathbf{U} + \frac{1}{b} (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+3}) \mathbf{D} \mathbf{U}. \end{aligned} \quad (4.19)$$

Similarly, from (4.13) and (4.15) we have

$$\begin{aligned} \mathbf{d}_n^{(2)} &= \tilde{\mathbf{e}}_n = \mathbf{e}_n - (\tilde{\mathbf{v}}_n + \tilde{\mathbf{v}}_{n+1}) \mathbf{W} \\ &= \mathbf{c}_{2n+1} - \left(\frac{1}{b} \{ \mathbf{c}_{2n} - (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+1}) \mathbf{D} \} + \frac{1}{b} \{ \mathbf{c}_{2n+2} - (\mathbf{c}_{2n+1} + \mathbf{c}_{2n+3}) \mathbf{D} \} \right) \mathbf{W} \\ &= \mathbf{c}_{2n+1} \left(\mathbf{I}_2 + \frac{2}{b} \mathbf{D} \mathbf{W} \right) - \frac{1}{b} (\mathbf{c}_{2n} + \mathbf{c}_{2n+2}) \mathbf{W} + \frac{1}{b} (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+3}) \mathbf{D} \mathbf{W}. \end{aligned} \quad (4.20)$$

We get the nonzero coefficients \mathbf{P}_k , $\mathbf{Q}_k^{(1)}$, $\mathbf{Q}_k^{(2)}$ of $\mathbf{P}(\omega)$, $\mathbf{Q}^{(1)}(\omega)$, $\mathbf{Q}^{(2)}(\omega)$ by comparing (4.18), (4.19) and (4.20) with (4.10):

$$\begin{aligned} \mathbf{P}_0 &= \frac{2}{b} \mathbf{I}_2, \quad \mathbf{P}_{-1} = \mathbf{P}_1 = -\frac{2}{b} \mathbf{D}^T; \\ \mathbf{Q}_0^{(1)} &= \mathbf{Q}_2^{(1)} = -\frac{2}{b} \mathbf{U}^T, \quad \mathbf{Q}_1^{(1)} = 2 \left(\mathbf{I}_2 + \frac{2}{b} \mathbf{U}^T \mathbf{D}^T \right), \quad \mathbf{Q}_{-1}^{(1)} = \mathbf{Q}_3^{(1)} = \frac{2}{b} \mathbf{U}^T \mathbf{D}^T; \\ \mathbf{Q}_0^{(2)} &= \mathbf{Q}_2^{(2)} = -\frac{2}{b} \mathbf{W}^T, \quad \mathbf{Q}_1^{(2)} = 2 \left(\mathbf{I}_2 + \frac{2}{b} \mathbf{W}^T \mathbf{D}^T \right), \quad \mathbf{Q}_{-1}^{(2)} = \mathbf{Q}_3^{(2)} = \frac{2}{b} \mathbf{W}^T \mathbf{D}^T. \end{aligned} \quad (4.21)$$

Therefore, the analysis multifilter bank $\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}$ is

$$\mathbf{P}(\omega) = \frac{1}{2} \left(\frac{2}{b} \mathbf{I}_2 - \frac{2}{b} (e^{i\omega} + e^{-i\omega}) \mathbf{D}^T \right); \quad (4.22)$$

$$\mathbf{Q}^{(1)}(\omega) = \frac{1}{2} \left(e^{-i\omega} \left(2\mathbf{I}_2 + \frac{4}{b} \mathbf{U}^T \mathbf{D}^T \right) - \frac{2}{b} (1 + e^{-2i\omega}) \mathbf{U}^T + \frac{2}{b} (e^{i\omega} + e^{-3i\omega}) \mathbf{U}^T \mathbf{D}^T \right); \quad (4.23)$$

$$\mathbf{Q}^{(2)}(\omega) = \frac{1}{2} \left(e^{-i\omega} \left(2\mathbf{I}_2 + \frac{4}{b} \mathbf{W}^T \mathbf{D}^T \right) - \frac{2}{b} (1 + e^{-2i\omega}) \mathbf{W}^T + \frac{2}{b} (e^{i\omega} + e^{-3i\omega}) \mathbf{W}^T \mathbf{D}^T \right) \quad (4.24)$$

We will obtain the synthesis multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ from (4.12) and (4.16) we have

$$\begin{aligned}
\mathbf{c}_{2k+1} &= \mathbf{e}_k = t\{\tilde{\mathbf{f}}_k + (\tilde{\mathbf{v}}_k + \tilde{\mathbf{v}}_{k+1})\mathbf{U}\} + (1-t)\{\tilde{\mathbf{e}}_k + (\tilde{\mathbf{v}}_k + \tilde{\mathbf{v}}_{k+1})\mathbf{W}\} \\
&= t\{\mathbf{d}_k^{(1)} + (\tilde{\mathbf{c}}_k + \tilde{\mathbf{c}}_{k+1})\mathbf{U}\} + (1-t)\{\mathbf{d}_k^{(2)} + (\tilde{\mathbf{c}}_k + \tilde{\mathbf{c}}_{k+1})\mathbf{W}\}.
\end{aligned} \tag{4.25}$$

From (4.12) and (4.17)

$$\begin{aligned}
\mathbf{c}_{2k} &= \mathbf{v}_k = b\tilde{\mathbf{v}}_k + (\mathbf{e}_{k-1} + \mathbf{e}_k)\mathbf{D} \\
&= b\tilde{\mathbf{c}}_k + (t\{\mathbf{d}_{k-1}^{(1)} + (\tilde{\mathbf{c}}_{k-1} + \tilde{\mathbf{c}}_k)\mathbf{U}\} + (1-t)\{\mathbf{d}_{k-1}^{(2)} + (\tilde{\mathbf{c}}_{k-1} + \tilde{\mathbf{c}}_k)\mathbf{W}\} + t\{\mathbf{d}_k^{(1)} \\
&\quad + (\tilde{\mathbf{c}}_k + \tilde{\mathbf{c}}_{k+1})\mathbf{U}\} + (1-t)\{\mathbf{d}_k^{(2)} + (\tilde{\mathbf{c}}_k + \tilde{\mathbf{c}}_{k+1})\mathbf{W}\})\mathbf{D} \\
&= \tilde{\mathbf{c}}_k (b\mathbf{I}_2 + 2t\mathbf{UD} + 2(1-t)\mathbf{WD}) + (\tilde{\mathbf{c}}_{k-1} + \tilde{\mathbf{c}}_{k+1}) (t\mathbf{UD} + (1-t)\mathbf{WD}) \\
&\quad + t(\mathbf{d}_{k-1}^{(1)} + \mathbf{d}_k^{(1)})\mathbf{D} + (1-t)(\mathbf{d}_{k-1}^{(2)} + \mathbf{d}_k^{(2)})\mathbf{D}.
\end{aligned} \tag{4.26}$$

By comparing (4.25) and (4.11) for odd k , we have that the nonzero coefficients \mathbf{P}_{2k+1} , $\mathbf{Q}_{2k+1}^{(1)}$, $\mathbf{Q}_{2k+1}^{(2)}$ with odd indices $2k+1$ are

$$\begin{aligned}
\tilde{\mathbf{P}}_1 &= \tilde{\mathbf{P}}_{-1} = t\mathbf{U} + (1-t)\mathbf{W}; \\
\tilde{\mathbf{Q}}_1^{(1)} &= t\mathbf{I}_2; \\
\tilde{\mathbf{Q}}_1^{(2)} &= (1-t)\mathbf{I}_2,
\end{aligned} \tag{4.27}$$

where the other coefficients are zero.

Now, we compare (4.26) and (4.11) for even k , we have that the nonzero coefficients \mathbf{P}_{2k} , $\mathbf{Q}_{2k}^{(1)}$, $\mathbf{Q}_{2k}^{(2)}$ with even indices $2k$ are

$$\begin{aligned}
\tilde{\mathbf{P}}_0 &= b\mathbf{I}_2 + 2t\mathbf{UD} + 2(1-t)\mathbf{WD}, \quad \tilde{\mathbf{P}}_2 = \tilde{\mathbf{P}}_{-2} = t\mathbf{UD} + (1-t)\mathbf{WD}; \\
\tilde{\mathbf{Q}}_0^{(1)} &= \tilde{\mathbf{Q}}_2^{(1)} = t\mathbf{D}; \\
\tilde{\mathbf{Q}}_0^{(2)} &= \tilde{\mathbf{Q}}_2^{(2)} = (1-t)\mathbf{D}.
\end{aligned} \tag{4.28}$$

Hence the synthesis multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ is

$$\begin{aligned}
\tilde{\mathbf{P}}(\omega) &= \frac{1}{2}(b\mathbf{I}_2 + 2t\mathbf{UD} + 2(1-t)\mathbf{WD}) + (e^{i\omega} + e^{-i\omega})(t\mathbf{U} + (1-t)\mathbf{W}) \\
&\quad + (e^{2i\omega} + e^{-2i\omega})(t\mathbf{UD} + (1-t)\mathbf{WD});
\end{aligned} \tag{4.29}$$

$$\tilde{\mathbf{Q}}^{(1)}(\omega) = \frac{1}{2}(t(1 + e^{-2i\omega})\mathbf{D}) + te^{-i\omega}\mathbf{I}_2; \tag{4.30}$$

$$\tilde{\mathbf{Q}}^{(2)}(\omega) = \frac{1}{2}((1-t)(1 + e^{-2i\omega})\mathbf{D}) + (1-t)e^{-i\omega}\mathbf{I}_2. \tag{4.31}$$

Denote

$$\left\{ \begin{array}{l} \mathbf{C}_1(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -(1 + e^{-i\omega})\mathbf{U}^T & \mathbf{I}_2 & \mathbf{0}_2 \\ -(1 + e^{-i\omega})\mathbf{W}^T & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{6 \times 6} \\ \mathbf{C}_0(\omega) = \begin{bmatrix} \frac{1}{b}\mathbf{I}_2 & -\frac{1}{b}(1 + e^{i\omega})\mathbf{D}^T \\ \mathbf{0}_2 & \mathbf{I}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{6 \times 4} \end{array} \right. , \quad (4.32)$$

and $\tilde{\mathbf{C}}_1(\omega) = (\mathbf{C}_1(\omega)^{-1})^*$ and $\tilde{\mathbf{C}}_0(\omega) = (\mathbf{C}_0(\omega)^{-1})^*$ are given by

$$\left\{ \begin{array}{l} \tilde{\mathbf{C}}_1(\omega) = \begin{bmatrix} \mathbf{I}_2 & (1 + e^{i\omega})\mathbf{U} & (1 + e^{i\omega})\mathbf{W} \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{6 \times 6} \\ \tilde{\mathbf{C}}_0(\omega) = \begin{bmatrix} b\mathbf{I}_2 & \mathbf{0}_2 \\ t(1 + e^{-i\omega})\mathbf{D} & t\mathbf{I}_2 \\ (1 - t)(1 + e^{-i\omega})\mathbf{D} & (1 - t)\mathbf{I}_2 \end{bmatrix}_{6 \times 4} \end{array} \right. . \quad (4.33)$$

Then, the multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ corresponding to this 2-step frame algorithm can be written as

$$\begin{bmatrix} \mathbf{P}(\omega) \\ \mathbf{Q}^{(1)}(\omega) \\ \mathbf{Q}^{(2)}(\omega) \end{bmatrix}_{6 \times 2} = \mathbf{C}_1(2\omega)\mathbf{C}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}_{4 \times 2} , \quad (4.34)$$

and

$$\begin{bmatrix} \tilde{\mathbf{P}}(\omega) \\ \tilde{\mathbf{Q}}^{(1)}(\omega) \\ \tilde{\mathbf{Q}}^{(2)}(\omega) \end{bmatrix}_{6 \times 2} = \frac{1}{2}\tilde{\mathbf{C}}_1(2\omega)\tilde{\mathbf{C}}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}_{4 \times 2} . \quad (4.35)$$

By solving the system of equations for a sum rule order one of both \mathbf{P} , $\tilde{\mathbf{P}}$ and for a vanishing moment order one of $\mathbf{Q}^{(1)}$, $\mathbf{Q}^{(2)}$, we have 4 free parameters which are g , t , w_{12} , w_{22} .

$$a = 0, \quad b = 2, \quad c = 0, \quad d_{11} = \frac{-1}{2},$$

$$d_{12} = \frac{3}{2g}, \quad d_{21} = 0, \quad d_{22} = \frac{-7}{2}, \quad h = \frac{-1}{3}, \quad u_{11} = \frac{1}{2}, \quad u_{12} = w_{12}(t - 1)/t, \quad u_{21} = 0,$$

$$u_{22} = (\frac{1}{8}(8t w_{22} - 8w_{22} + 1))/t, \quad w_{11} = \frac{1}{2}, \quad w_{21} = 0.$$

If we choose

$$[g, t, w_{12}, w_{22}] = \left[\frac{321}{32}, \frac{1763}{17}, \frac{-4217}{89}, \frac{-1613}{65} \right].$$

Using the smoothness formula, we can obtain $\tilde{\Phi} \in W^{0.3547}$ and it is supported on $[-2, 2]$ and Φ is supported on $[-2, 2]$, but we cannot find $\Phi \in L_2(\mathbb{R})$.

However, we will show the lowpass analysis mask \mathbf{P} and the lowpass synthesis mask $\tilde{\mathbf{P}}$ which are corresponding to this algorithm.

$$\mathbf{P}(\omega) = \begin{bmatrix} 0.25(e^{i\omega} + e^{-i\omega}) + 0.5 & 0 \\ -0.0747663551401(e^{i\omega} + e^{-i\omega}) & 1.75(e^{i\omega} + e^{-i\omega}) + 0.5 \end{bmatrix},$$

and

$$\tilde{\mathbf{P}}(\omega) = \begin{bmatrix} \tilde{P}_{11}(\omega) & \tilde{P}_{12}(\omega) \\ \tilde{P}_{21}(\omega) & \tilde{P}_{22}(\omega) \end{bmatrix},$$

where

$$\begin{aligned} \tilde{P}_{11}(\omega) &= -0.125(e^{2i\omega} + e^{-2i\omega}) + 0.75 + 0.75(e^{i\omega} + e^{-i\omega}), \\ \tilde{P}_{12}(\omega) &= 0, \\ \tilde{P}_{21}(\omega) &= 0.0093457943925(e^{2i\omega} + e^{-2i\omega}) + 0.0186915887850, \\ \tilde{P}_{22}(\omega) &= -0.21875(e^{2i\omega} + e^{-2i\omega}) + 0.5625 + 0.0625(e^{i\omega} + e^{-i\omega}). \end{aligned}$$

Thus, the resulting Φ is the linear B -spline supported on $[-1, 1]$.

The highpass analysis mask $\mathbf{Q}^{(\ell)}$ and the highpass synthesis mask $\tilde{\mathbf{Q}}^{(\ell)}$ will be as follows:

$$\mathbf{Q}_1^{(1)}(\omega) = \begin{bmatrix} Q_{11}^{(1)}(\omega) & Q_{12}^{(1)}(\omega) \\ Q_{21}^{(1)}(\omega) & Q_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned} Q_{11}^{(1)}(\omega) &= -0.125(e^{i\omega} + e^{-3i\omega}) - 0.25(1 + e^{-2i\omega}) + 0.75e^{-i\omega}, \\ Q_{12}^{(1)}(\omega) &= 0, \\ Q_{21}^{(1)}(\omega) &= 9.89390831396(e^{i\omega} + e^{-3i\omega}) + 23.4625669982856(1 + e^{-2i\omega}) + 19.78781662792e^{-i\omega}, \\ Q_{22}^{(1)}(\omega) &= 43.0060629281382(e^{i\omega} + e^{-3i\omega}) + 12.2874465508966(1 + e^{-2i\omega}) + 87.0121258562764e^{-i\omega}. \end{aligned}$$

$$\mathbf{Q}_2^{(2)}(\omega) = \begin{bmatrix} Q_{11}^{(2)}(\omega) & Q_{12}^{(2)}(\omega) \\ Q_{21}^{(2)}(\omega) & Q_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}
Q_{11}^{(2)}(\omega) &= -0.125(e^{i\omega} + e^{-3i\omega}) - 0.25(1 + e^{-2i\omega}) + 0.75e^{-i\omega}, \\
Q_{12}^{(2)}(\omega) &= 0, \\
Q_{21}^{(2)}(\omega) &= 9.89390831396(e^{i\omega} + e^{-3i\omega}) + 23.4625669982856(1 + e^{-2i\omega}) + 19.78781662792e^{-i\omega}, \\
Q_{22}^{(2)}(\omega) &= 43.0060629281382(e^{i\omega} + e^{-3i\omega}) + 12.2874465508966(1 + e^{-2i\omega}) + 87.0121258562764e^{-i\omega}.
\end{aligned}$$

$$\tilde{\mathbf{Q}}^{(1)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(1)}(\omega) & \tilde{Q}_{12}^{(1)}(\omega) \\ \tilde{Q}_{21}^{(1)}(\omega) & \tilde{Q}_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}
\tilde{Q}_{11}^{(1)}(\omega) &= -20.5e^{-2i\omega} - 25.9264705882352(1 + e^{-i\omega}), \\
\tilde{Q}_{12}^{(1)}(\omega) &= 7.6789444749862(1 + e^{-2i\omega}), \\
\tilde{Q}_{21}^{(1)}(\omega) &= 0, \\
\tilde{Q}_{22}^{(1)}(\omega) &= -181.4852941176470(1 + e^{-2i\omega}) + 51.8529411764705e^{-i\omega}.
\end{aligned}$$

$$\tilde{\mathbf{Q}}^{(2)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(2)}(\omega) & \tilde{Q}_{12}^{(2)}(\omega) \\ \tilde{Q}_{21}^{(2)}(\omega) & \tilde{Q}_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}
\tilde{Q}_{11}^{(2)}(\omega) &= 25.6764705882352(1 + e^{-2i\omega}) - 51.3529411764705e^{-i\omega}, \\
\tilde{Q}_{12}^{(2)}(\omega) &= -7.6789444749862(1 + e^{-2i\omega}), \\
\tilde{Q}_{21}^{(2)}(\omega) &= 0, \\
\tilde{Q}_{22}^{(2)}(\omega) &= 179.7352941176470(1 + e^{-2i\omega}) - 51.3529411764705e^{-i\omega}.
\end{aligned}$$

We consider Sobolev smoothness when we consider the smoothness of multiwavelets/multiple frames. In this work we have found the parameters for each algorithm by using Maple we solve the system of linear equations after we obtain the sum rule orders of lowpass multi-filter and the vanishing moments of highpass multifilters. The algorithms that we use will determine the orders of sum rule and vanishing moments. However, For each algorithm, when we use the obtained parameters to construct multiple bi-frames, we should select the values of the free parameters such that the synthesis multiscaling function $\tilde{\Phi}$ is smoother than the analysis multiscaling function Φ . Also, the analysis highpass multifilters should have higher vanishing moments. That means $\tilde{\mathbf{P}}$ has a higher sum rule order than \mathbf{P} . Thus,

$\mathbf{Q}^{(\ell)}$ has higher vanishing moment order than $\tilde{\mathbf{Q}}^{(\ell)}$. By minimization (using Matlab), we can select the remaining parameters such that the multiscaling function Φ and/or its dual $\tilde{\Phi}$ have optimal smoothness.

In the next subsection we consider a 3-step algorithm.

4.2.2 3-Step Type I Multiple Bi-frame Algorithm

The decomposition algorithm of this 3-step algorithm is given in (4.36)-(4.38) as shown in Fig. 4.3. In Step 1 given by formula (4.36), we replace each \mathbf{v} associated with an even node $2k$ by \mathbf{v}'' . Then, with the obtained \mathbf{v}'' , we obtain \mathbf{f}'' and \mathbf{e}'' both are associated with odd nodes $2k+1$ by (4.37). Finally, with the obtained \mathbf{v}'' in Step 1, we replace it by $\tilde{\mathbf{v}}$ in (4.38).

3-step Type I Multiple Bi-frame Decomposition Algorithm:

$$\text{Step 1. } \mathbf{v}'' = \{\mathbf{v} - (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{D}\} \mathbf{B}^{-1}; \quad (4.36)$$

$$\text{Step 2. } \begin{cases} \mathbf{f}'' = \mathbf{e} - (\mathbf{v}_0'' + \mathbf{v}_1'') \mathbf{U}, \\ \mathbf{e}'' = \mathbf{e} - (\mathbf{v}_0'' + \mathbf{v}_1'') \mathbf{W}; \end{cases} \quad (4.37)$$

$$\text{Step 3. } \tilde{\mathbf{v}} = \mathbf{v}'' - (\mathbf{f}_{-1}'' + \mathbf{f}_0'') \mathbf{D}_1 - (\mathbf{f}_{-2}'' + \mathbf{f}_1'') \mathbf{C}_1 - (\mathbf{e}_{-1}'' + \mathbf{e}_0'') \mathbf{N}_1 - (\mathbf{e}_{-2}'' + \mathbf{e}_1'') \mathbf{M}_1. \quad (4.38)$$

3-step Type I Multiple Bi-frame Reconstruction Algorithm:

$$\text{Step 1. } \mathbf{v}'' = \tilde{\mathbf{v}} + (\mathbf{f}_{-1}'' + \mathbf{f}_0'') \mathbf{D}_1 + (\mathbf{f}_{-2}'' + \mathbf{f}_1'') \mathbf{C}_1 + (\mathbf{e}_{-1}'' + \mathbf{e}_0'') \mathbf{N}_1 + (\mathbf{e}_{-2}'' + \mathbf{e}_1'') \mathbf{M}_1; \quad (4.39)$$

$$\text{Step 2. } \mathbf{e} = \{\mathbf{f}'' + (\mathbf{v}_0'' + \mathbf{v}_1'') \mathbf{U}\} \mathbf{T} + \{\mathbf{e}'' + (\mathbf{v}_0'' + \mathbf{v}_1'') \mathbf{W}\} (\mathbf{I}_2 - \mathbf{T}); \quad (4.40)$$

$$\text{Step 3. } \mathbf{v} = \mathbf{v}'' \mathbf{B} + (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{D}, \quad (4.41)$$

where \mathbf{B} , \mathbf{D} , \mathbf{U} , \mathbf{W} , \mathbf{D}_1 , \mathbf{C}_1 , \mathbf{N}_1 , \mathbf{M}_1 , and \mathbf{T} are 2×2 matrices and the entries of these matrices are some constants in \mathbb{R} .

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix},$$

$$\mathbf{D}_1 = \begin{bmatrix} d_{1,11} & d_{1,12} \\ d_{1,21} & d_{1,22} \end{bmatrix}, \quad \mathbf{C}_1 = \begin{bmatrix} c_{1,11} & c_{1,12} \\ c_{1,21} & c_{1,22} \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} n_{1,11} & n_{1,12} \\ n_{1,21} & n_{1,22} \end{bmatrix}, \quad \mathbf{M}_1 = \begin{bmatrix} m_{1,11} & m_{1,12} \\ m_{1,21} & m_{1,22} \end{bmatrix},$$

$$\text{and } \mathbf{T} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}.$$

Fig. 4.4 shows that the backward algorithm of the decomposition algorithm is the reconstruction algorithm. The multiresolution reconstruction algorithm is given in (4.39)-(

4.41), where the matrices are the same 2×2 matrices in the decomposition algorithm.

First in Step 1, we update $\tilde{\mathbf{v}}$ of the lowpass output by \mathbf{v}'' given in (4.39). Then, in Step 2 with the obtained \mathbf{v}'' in Step 1, we obtain \mathbf{e} given by formula (4.40). Finally, we replace \mathbf{v}'' which is obtained in Step 1 by \mathbf{v} given in (4.41). So the inputs here are $\tilde{\mathbf{v}}$, \mathbf{f}'' , \mathbf{e}'' and the outputs are \mathbf{v}'' , \mathbf{e} , \mathbf{v} .

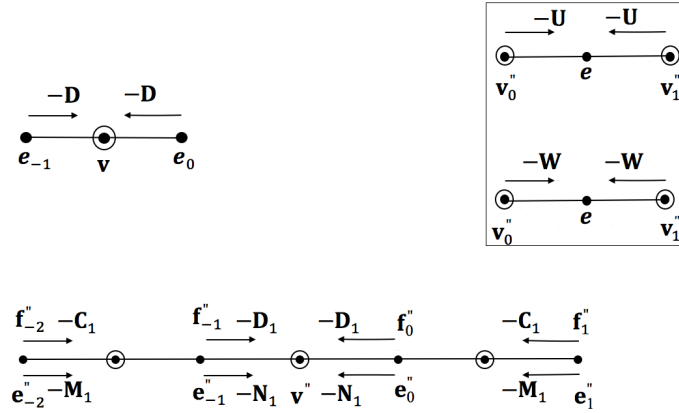


Figure 4.3: Top-left: Decomposition Step 1; Top-right: Decomposition Step 2; bottom: Decomposition Step 3.

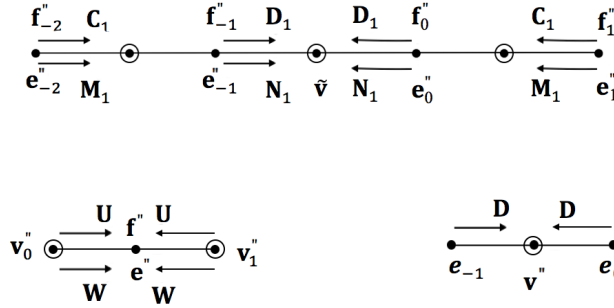


Figure 4.4: Top: Reconstruction Step 1; bottom-left: Reconstruction Step 2; bottom-right: Reconstruction Step 3.

After following the same calculation details in 2-step frame algorithm, we can obtain

the analysis multifilter bank

$$\begin{aligned}
\mathbf{P}(\omega) = & \frac{1}{2}(2(\mathbf{B}^{-1})^T + 4(\mathbf{D}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T) + 4(\mathbf{N}_1^T \mathbf{W}^T (\mathbf{B}^{-1})^T) + (e^{i\omega} + e^{-i\omega}) \\
& (-2(\mathbf{B}^{-1})^T \mathbf{D}^T - 6(\mathbf{D}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T) - 2(\mathbf{C}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T) - 6(\mathbf{N}_1^T \mathbf{W}^T \\
& (\mathbf{B}^{-1})^T \mathbf{D}^T) - 2(\mathbf{M}_1^T \mathbf{W}^T (\mathbf{B}^{-1})^T \mathbf{D}^T) - 2\mathbf{D}_1^T - 2\mathbf{N}_1^T) + (e^{2i\omega} + e^{-2i\omega})(2\mathbf{D}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T \\
& + 2\mathbf{C}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T + 2\mathbf{N}_1^T \mathbf{W}^T (\mathbf{B}^{-1})^T + 2(\mathbf{M}_1^T \mathbf{W}^T)(\mathbf{B}^{-1})^T) \\
& + (e^{3i\omega} + e^{-3i\omega})(-2(\mathbf{D}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T) - 2(\mathbf{N}_1^T \mathbf{W}^T (\mathbf{B}^{-1})^T \mathbf{D}^T) \\
& - 4(\mathbf{C}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T) - 4(\mathbf{M}_1^T \mathbf{W}^T (\mathbf{B}^{-1})^T \mathbf{D}^T) - 2\mathbf{M}_1^T - 2\mathbf{C}_1^T) \\
& + (e^{4i\omega} + e^{-4i\omega})(2\mathbf{C}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T + 2\mathbf{M}_1^T \mathbf{W}^T (\mathbf{B}^{-1})^T) \\
& + (e^{5i\omega} + e^{-5i\omega})(-2\mathbf{C}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T - 2\mathbf{M}_1^T \mathbf{W}^T (\mathbf{B}^{-1})^T \mathbf{D}^T)); \tag{4.42}
\end{aligned}$$

$$\begin{aligned}
\mathbf{Q}^{(1)}(\omega) = & \frac{1}{2}(-2\mathbf{U}^T (\mathbf{B}^{-1})^T + e^{i\omega}(4\mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T + 2\mathbf{I}_2) + e^{-i\omega}(2\mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T); \\
& + e^{-2i\omega}(-2\mathbf{U}^T (\mathbf{B}^{-1})^T) + e^{-3i\omega}(2\mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T)) \tag{4.43}
\end{aligned}$$

$$\begin{aligned}
\mathbf{Q}^{(2)}(\omega) = & \frac{1}{2}(-2\mathbf{W}^T (\mathbf{B}^{-1})^T + e^{i\omega}(4\mathbf{W}^T (\mathbf{B}^{-1})^T \mathbf{D}^T + 2\mathbf{I}_2) + e^{-i\omega}(2\mathbf{W}^T (\mathbf{B}^{-1})^T \mathbf{D}^T) \\
& + e^{-2i\omega}(-2\mathbf{W}^T (\mathbf{B}^{-1})^T) + e^{-3i\omega}(2\mathbf{W}^T (\mathbf{B}^{-1})^T \mathbf{D}^T)). \tag{4.44}
\end{aligned}$$

and the synthesis multifilter bank is

$$\begin{aligned}
\tilde{\mathbf{P}}(\omega) = & \frac{1}{2}(\mathbf{B} + 2\mathbf{U}\mathbf{T}\mathbf{D} + 2\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D}) + (e^{i\omega} + e^{-i\omega})(\mathbf{U}\mathbf{T} + \mathbf{W}(\mathbf{I}_2 - \mathbf{T})) + (e^{2i\omega} + e^{-2i\omega}) \\
& (\mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D}); \tag{4.45}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{Q}}^{(1)}(\omega) = & \frac{1}{2}(\mathbf{D}_1\mathbf{B} + 3\mathbf{D}_1\mathbf{U}\mathbf{T}\mathbf{D} + 3\mathbf{D}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D} + \mathbf{C}_1\mathbf{U}\mathbf{T}\mathbf{D} \\
& + \mathbf{T}\mathbf{D} + \mathbf{C}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D} + e^{i\omega}(\mathbf{D}_1\mathbf{U}\mathbf{T} + \mathbf{C}_1\mathbf{U}\mathbf{T} + \mathbf{D}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T}) \\
& + \mathbf{C}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})) + e^{-i\omega}(\mathbf{T} + 2\mathbf{D}_1\mathbf{U}\mathbf{T} + 2\mathbf{D}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})) + e^{2i\omega}(\mathbf{D}_1\mathbf{B} \\
& + \mathbf{D}_1\mathbf{U}\mathbf{T}\mathbf{D} + 2\mathbf{C}_1\mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{D}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D} + 2\mathbf{C}_1\mathbf{W}(\mathbf{I}_2 \\
& - \mathbf{T})\mathbf{D} + \mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{D}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T}) + e^{-2i\omega}(\mathbf{D}_1\mathbf{B} + 3\mathbf{D}_1\mathbf{U}\mathbf{T}\mathbf{D} + 3\mathbf{D}_1\mathbf{W} \\
& (\mathbf{I}_2 - \mathbf{T})\mathbf{D} + \mathbf{C}_1\mathbf{U}\mathbf{T}\mathbf{D} + 2\mathbf{T}\mathbf{D} + \mathbf{C}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D} + \mathbf{D}_1\mathbf{T}) + e^{3i\omega}(\mathbf{D}_1 \\
& \mathbf{U}\mathbf{T} + \mathbf{C}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})) + e^{-3i\omega}(\mathbf{N}_1\mathbf{U}\mathbf{T} + 2\mathbf{M}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T}) + \mathbf{M}_1\mathbf{U}\mathbf{T} \\
& + \mathbf{N}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})) + e^{-4i\omega}(\mathbf{M}_1\mathbf{B} + \mathbf{N}_1\mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{N}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D} \\
& + 2\mathbf{M}_1\mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{M}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D}) + e^{4i\omega}(2\mathbf{C}_1\mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{C}_1\mathbf{W}(\mathbf{I}_2 \\
& - \mathbf{T})\mathbf{D}) + e^{-5i\omega}(2\mathbf{C}_1\mathbf{U}\mathbf{T} + \mathbf{C}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})) + e^{-6i\omega}(\mathbf{C}_1\mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{C}_1\mathbf{W} \\
& (\mathbf{I}_2 - \mathbf{T})\mathbf{D}); \tag{4.46}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{Q}}^{(2)}(\omega) = & \frac{1}{2}(\mathbf{N}_1\mathbf{B} + 3\mathbf{N}_1\mathbf{U}\mathbf{T}\mathbf{D} + 3\mathbf{N}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D} + \mathbf{M}_1\mathbf{U}\mathbf{T}\mathbf{D} + (\mathbf{I}_2 - \mathbf{T})\mathbf{D} \\
& + \mathbf{M}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D}) + e^{i\omega}(\mathbf{N}_1\mathbf{U}\mathbf{T} + \mathbf{M}_1\mathbf{U}\mathbf{T} + \mathbf{N}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T}) + \mathbf{M}_1\mathbf{W} \\
& (\mathbf{I}_2 - \mathbf{T})) + e^{-i\omega}((\mathbf{I}_2 - \mathbf{T}) + 2\mathbf{N}_1\mathbf{U}\mathbf{T} + 2\mathbf{N}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})) + e^{2i\omega}(\mathbf{N}_1\mathbf{B} + \mathbf{N}_1\mathbf{U} \\
& \mathbf{T}\mathbf{D} + 2\mathbf{M}_1\mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{N}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D} + 2\mathbf{M}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D} + \mathbf{U}\mathbf{T}\mathbf{D} \\
& + \mathbf{N}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})) + e^{-2i\omega}(\mathbf{N}_1\mathbf{B} + 3\mathbf{N}_1\mathbf{U}\mathbf{T}\mathbf{D} + 3\mathbf{N}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D} \\
& + \mathbf{M}_1\mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{T}\mathbf{D} + \mathbf{M}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D} + \mathbf{N}_1\mathbf{T} + (\mathbf{I}_2 - \mathbf{T})\mathbf{D}) + e^{3i\omega}(\mathbf{N}_1 \\
& \mathbf{U}\mathbf{T} + \mathbf{M}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})) + e^{-3i\omega}(\mathbf{N}_1\mathbf{U}\mathbf{T} + \mathbf{M}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T}) + \mathbf{M}_1\mathbf{U}\mathbf{T} + \mathbf{N}_1\mathbf{W} \\
& (\mathbf{I}_2 - \mathbf{T})) + e^{-4i\omega}(\mathbf{M}_1\mathbf{B} + \mathbf{N}_1\mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{N}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D} + 2\mathbf{M}_1\mathbf{U}\mathbf{T}\mathbf{D} \\
& + \mathbf{M}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D})) + e^{4i\omega}(2\mathbf{M}_1\mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{M}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D}) + e^{-5i\omega}(\mathbf{M}_1 \\
& \mathbf{U}\mathbf{T} + \mathbf{M}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})) + e^{-6i\omega}(\mathbf{M}_1\mathbf{U}\mathbf{T}\mathbf{D} + \mathbf{M}_1\mathbf{W}(\mathbf{I}_2 - \mathbf{T})\mathbf{D}). \tag{4.47}
\end{aligned}$$

Denote

$$\left\{ \begin{aligned} \mathbf{B}_2(2\omega) &= \begin{bmatrix} \mathbf{I}_2 & -(1 + e^{2i\omega})\mathbf{D}_1^T - (e^{-2i\omega} + e^{4i\omega})\mathbf{C}_1^T & -(1 + e^{2i\omega})\mathbf{N}_1^T - (e^{-2i\omega} + e^{4i\omega})\mathbf{M}_1^T \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{6 \times 6}, \\ \mathbf{B}_1(2\omega) &= \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -(1 + e^{-2i\omega})\mathbf{U}^T & \mathbf{I}_2 & \mathbf{0}_2 \\ -(1 + e^{-2i\omega})\mathbf{W}^T & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{6 \times 6}, \\ \mathbf{B}_0(2\omega) &= \begin{bmatrix} (\mathbf{B}^{-1})^T & -(1 + e^{2i\omega})(\mathbf{B}^{-1})^T\mathbf{D}^T \\ \mathbf{0}_2 & \mathbf{I}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{6 \times 4}, \end{aligned} \right. \tag{4.48}$$

where $\mathbf{B}_1 = \mathbf{C}_1$ as defined in (4.32), and

$$\left\{ \begin{aligned} \tilde{\mathbf{B}}_2(2\omega) &= \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1 + e^{-2i\omega})\mathbf{D}_1 + (e^{2i\omega} + e^{-4i\omega})\mathbf{C}_1 & \mathbf{I}_2 & \mathbf{0}_2 \\ (1 + e^{-2i\omega})\mathbf{N}_1 + (e^{2i\omega} + e^{-4i\omega})\mathbf{M}_1 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{6 \times 6}, \\ \tilde{\mathbf{B}}_1(2\omega) &= \begin{bmatrix} \mathbf{I}_2 & (1 + e^{2i\omega})\mathbf{U} & (1 + e^{2i\omega})\mathbf{W} \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{6 \times 6}, \\ \tilde{\mathbf{B}}_0(2\omega) &= \begin{bmatrix} \mathbf{B} & \mathbf{0}_2 \\ (1 + e^{-2i\omega})\mathbf{T}\mathbf{D} & \mathbf{T} \\ (1 + e^{-2i\omega})(\mathbf{I}_2 - \mathbf{T})\mathbf{D} & (\mathbf{I}_2 - \mathbf{T}) \end{bmatrix}_{6 \times 4}. \end{aligned} \right. \quad (4.49)$$

where $\tilde{\mathbf{B}}_1 = \tilde{\mathbf{C}}_1$ as defined in (4.33). Then $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ can be written as

$$\begin{bmatrix} \mathbf{P}(\omega) \\ \mathbf{Q}^{(1)}(\omega) \\ \mathbf{Q}^{(2)}(\omega) \end{bmatrix}_{6 \times 2} = \mathbf{B}_2(2\omega)\mathbf{B}_1(2\omega)\mathbf{B}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}_{4 \times 2}, \quad (4.50)$$

and

$$\begin{bmatrix} \tilde{\mathbf{P}}(\omega) \\ \tilde{\mathbf{Q}}^{(1)}(\omega) \\ \tilde{\mathbf{Q}}^{(2)}(\omega) \end{bmatrix}_{6 \times 2} = \frac{1}{2}\tilde{\mathbf{B}}_2(2\omega)\tilde{\mathbf{B}}_1(2\omega)\tilde{\mathbf{B}}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}_{4 \times 2}. \quad (4.51)$$

Similar to 2-step algorithm, we can find the parameters after solving the system of equations for a sum rule order of both \mathbf{P} and $\tilde{\mathbf{P}}$ and for a vanishing moment order of both $\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}$.

First, we solve the system of equations for a sum rule order one of $\tilde{\mathbf{P}}$, for a sum rule order one of \mathbf{P} and for a vanishing moment order one of both $\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}$. Then, we have 25 free parameters which are

$$b_{21}, d_{1,11}, d_{1,21}, d_{11}, d_{12}, d_{21}, d_{22}, m_{11}, m_{21}, n_{11}, n_{21}, t_{11}, t_{12}, t_{21}, t_{22}, \\ u_{22}, w_{21}, d_{1,12}, d_{1,22}, n_{12}, n_{22}, m_{12}, m_{22}, c_{12}, c_{22}.$$

and the other parameters are

$$b_{11} = -2d_{11} + 1, b_{12} = -2d_{12}, b_{22} = ((1/2)(8d_{11}d_{22}w_{21} - 8d_{12}d_{21}w_{21} + 2b_{21}d_{22} - 4d_{11}w_{21} - 4d_{22}w_{21} - b_{21} + 2w_{21}))/d_{21}, c_{11} = -d_{1,11} - \frac{1}{2}d_{11} - m_{11} - n_{11} - \frac{1}{4}, c_{21} = -d_{1,21} - \frac{1}{2}d_{21} -$$

$$m_{21} - n_{21}, u_{11} = \frac{1}{2}, u_{12} = 0, u_{21} = w_{21}, w_{11} = \frac{1}{2}, w_{12} = 0, w_{22} = (\frac{1}{8}(4d_{11}w_{21} + 4d_{21}u_{22} + 8m_{21}u_{22} + 8n_{21}u_{22} + b_{21} + 2w_{21}))/ (m_{21} + n_{21}).$$

If we choose

$$\begin{aligned} & [b_{21}, d_{1,11}, d_{1,21}, d_{11}, d_{12}, d_{21}, d_{22}, m_{11}, m_{21}, n_{11}, n_{21}, t_{11}, t_{12}, t_{21}, t_{22}, u_{22}, w_{21}, d_{1,12}, \\ & d_{1,22}, n_{12}, n_{22}, m_{12}, m_{22}, c_{12}, c_{22}] \\ = & \left[\frac{-41}{2377}, \frac{-41}{191}, \frac{7}{860}, \frac{186}{745}, \frac{-6}{349}, \frac{118}{425}, \frac{-3}{341}, \frac{365}{498}, \frac{242}{295}, \frac{-93}{329}, \frac{-59}{455}, \frac{-29}{434}, \frac{131}{402}, \frac{24}{335}, \frac{227}{268}, \frac{35}{2519}, \right. \\ & \left. \frac{2}{445}, \frac{18}{103}, \frac{125}{473}, \frac{301}{1552}, \frac{61}{620}, \frac{-30}{431}, \frac{137}{1321}, \frac{-71}{507}, \frac{-31}{293} \right]. \end{aligned}$$

By using the smoothness formula, we can obtain $\tilde{\Phi} \in W^2$ and it is supported on $[-2, 2]$. The resulting $\Phi \in W^{1.2673}$ is supported on $[-5, 5]$.

The lowpass analysis mask \mathbf{P} and the lowpass synthesis mask $\tilde{\mathbf{P}}$ will be as follows:

$$\mathbf{P}(\omega) = \begin{bmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} P_{11}(\omega) = & -0.02709590787(e^{5i\omega} + e^{-5i\omega}) + 0.1154429373(e^{4i\omega} + e^{-4i\omega}) - 0.06329361571(e^{3i\omega} + e^{-3i\omega}) \\ & - 0.3600251780(e^{2i\omega} + e^{-2i\omega}) + 0.3403895236(e^{i\omega} + e^{-i\omega}) + 0.9891644813, \end{aligned}$$

$$\begin{aligned} P_{12}(\omega) = & -0.03116910104(e^{5i\omega} + e^{-5i\omega}) + 0.1004007398(e^{4i\omega} + e^{-4i\omega}) + 0.08425108744(e^{3i\omega} + e^{-3i\omega}) \\ & + 0.08425108744(e^{2i\omega} + e^{-2i\omega}) + 0.05308198640(e^{i\omega} + e^{-i\omega}) + 0.2157818330, \end{aligned}$$

$$\begin{aligned} P_{21}(\omega) = & 0.04864699392(e^{5i\omega} + e^{-5i\omega}) - 0.2021164959(e^{4i\omega} + e^{-4i\omega}) + 0.2260180468(e^{3i\omega} + e^{-3i\omega}) \\ & + 0.1440757088(e^{2i\omega} + e^{-2i\omega}) + 0.2676135094(e^{i\omega} + e^{-i\omega}) - 0.9684755260, \end{aligned}$$

$$\begin{aligned} P_{22}(\omega) = & 0.05518861559(e^{5i\omega} + e^{-5i\omega}) + 0.1055321168(e^{4i\omega} + e^{-4i\omega}) + 0.01917082569(e^{3i\omega} + e^{-3i\omega}) \\ & + 0.2150198502(e^{2i\omega} + e^{-2i\omega}) + 0.08652960981(e^{i\omega} + e^{-i\omega}) + 24.82519385. \end{aligned}$$

and

$$\tilde{\mathbf{P}}(\omega) = \begin{bmatrix} \tilde{P}_{11}(\omega) & \tilde{P}_{12}(\omega) \\ \tilde{P}_{21}(\omega) & \tilde{P}_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} \tilde{P}_{11}(\omega) = & 0.06241610738(e^{2i\omega} + e^{-2i\omega}) + 0.3751677852 + 0.25(e^{i\omega} + e^{-i\omega}), \\ \tilde{P}_{12}(\omega) = & -0.004297994269(e^{2i\omega} + e^{-2i\omega}) + 0.008595988539, \\ \tilde{P}_{21}(\omega) = & 0.002803094710(e^{2i\omega} + e^{-2i\omega}) - 0.003018126945 + 0.002171602739(e^{i\omega} + e^{-i\omega}), \\ \tilde{P}_{22}(\omega) = & -0.0001966338518(e^{2i\omega} + e^{-2i\omega}) + 0.01968920057 + 0.007108613728(e^{i\omega} + e^{-i\omega}). \end{aligned}$$

The highpass analysis mask \mathbf{Q} and the highpass synthesis mask $\tilde{\mathbf{Q}}$ will be as follows:

$$\mathbf{Q}_1^{(1)}(\omega) = \begin{bmatrix} Q_{11}^{(1)}(\omega) & Q_{12}^{(1)}(\omega) \\ Q_{21}^{(1)}(\omega) & Q_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$Q_{11}^{(1)}(\omega) = 1.462585817e^{i\omega} + 0.2312929085(e^{-i\omega} + e^{-3i\omega}) - 0.9625858169(1 + e^{-2i\omega}),$$

$$Q_{12}^{(1)}(\omega) = 0.5252758515e^{i\omega} + 0.2626379258(e^{-i\omega} + e^{-3i\omega}) - 0.5252758515(1 + e^{-2i\omega}),$$

$$Q_{21}^{(1)}(\omega) = -0.02307665651e^{i\omega} - 0.01153832825(e^{-i\omega} + e^{-3i\omega}) + 0.02307665651(1 + e^{-2i\omega}),$$

$$Q_{22}^{(1)}(\omega) = 0.9812732318e^{i\omega} - 0.009363384108(e^{-i\omega} + e^{-3i\omega}) - 0.3360234803(1 + e^{-2i\omega}),$$

.

$$\mathbf{Q}_2^{(2)}(\omega) = \begin{bmatrix} Q_{11}^{(2)}(\omega) & Q_{12}^{(2)}(\omega) \\ Q_{21}^{(2)}(\omega) & Q_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$Q_{11}^{(2)}(\omega) = 1.462585817e^{i\omega} + 0.2312929085(e^{-i\omega} + e^{-3i\omega}) + 0.9625858169(1 + e^{-2i\omega}),$$

$$Q_{12}^{(2)}(\omega) = 0.5252758515e^{i\omega} + 0.2626379258(e^{-i\omega} + e^{-3i\omega}) - 0.5252758515(1 + e^{-2i\omega}),$$

$$Q_{21}^{(2)}(\omega) = -0.02658135765e^{i\omega} + 0.038292857(e^{-i\omega} + e^{-3i\omega}) + 0.07151551(1 + e^{-2i\omega}),$$

$$Q_{22}^{(2)}(\omega) = 0.0301536395e^{i\omega} + 0.22807083(e^{-i\omega} + e^{-3i\omega}) + 0.47605288(1 + e^{-2i\omega}).$$

and

$$\tilde{\mathbf{Q}}_1^{(1)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(1)}(\omega) & \tilde{Q}_{12}^{(1)}(\omega) \\ \tilde{Q}_{21}^{(1)}(\omega) & \tilde{Q}_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned} \tilde{Q}_{11}^{(1)}(\omega) = & -0.2945485173e^{-i\omega} + 0.09985745363 - 0.2741304957e^{-2i\omega} + 0.4130329658e^{2i\omega} \\ & + 0.07689863669e^{-6i\omega} + 0.4123937981e^{i\omega} + 0.0999198e^{-5i\omega} - 0.31516085452e^{4i\omega} \\ & + 0.0336757e^{-4i\omega} - 0.056213325e^{3i\omega} + 0.091833358339e^{-3i\omega}, \end{aligned}$$

$$\begin{aligned}\tilde{Q}_{12}^{(1)}(\omega) = & 0.07160266520 + 0.095248202e^{-i\omega} + 0.979195473e^{2i\omega} + 0.39031777e^{-6i\omega} \\ & + 0.0836323688e^{i\omega} - 0.05029424e^{-5i\omega} + 0.9823381602e^{-2i\omega} - 0.81451648296e^{4i\omega} \\ & + 0.061369347e^{-4i\omega} + 0.037034330e^{3i\omega} + 0.025819807e^{-3i\omega},\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{21}^{(1)}(\omega) = & 0.0519826014e^{-4i\omega} - 0.0742980831e^{3i\omega} + 0.052964046e^{i\omega} - 0.52198895049e^{-5i\omega} \\ & - 0.5307228e^{4i\omega} + 0.03499644e^{-i\omega} - 0.67647511e^{2i\omega} + 0.058892123e^{-6i\omega} \\ & + 0.0241363 + 0.7114867e^{-2i\omega} + 0.0387034e^{-3i\omega},\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{22}^{(1)}(\omega) = & 0.03607056309e^{3i\omega} - 0.915505735e^{-5i\omega} - 0.1857573e^{4i\omega} + 0.3588921238e^{-6i\omega} \\ & + 0.3616678e^{-4i\omega} + 0.00312662e^{2i\omega} + 0.79587829e^{i\omega} + 0.956389128e^{-2i\omega} \\ & + 0.41363523 + 0.46152831e^{-i\omega} + 0.8701275e^{-3i\omega}.\end{aligned}$$

$$\tilde{\mathbf{Q}}_2^{(2)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(2)}(\omega) & \tilde{Q}_{12}^{(2)}(\omega) \\ \tilde{Q}_{21}^{(2)}(\omega) & \tilde{Q}_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}\tilde{Q}_{11}^{(2)}(\omega) = & 0.038189143(e^{i\omega} + e^{-3i\omega}) + 0.58346465144e^{3i\omega} + 0.1406822921e^{-6i\omega} \\ & + 0.0589404684e^{-5i\omega} + 0.94728698 + 0.671748e^{-2i\omega} + 0.05302545e^{-i\omega} \\ & + 0.0857346142e^{2i\omega} + 0.78734947535e^{4i\omega} + 0.660420410e^{-4i\omega},\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{12}^{(2)}(\omega) = & -0.287014536 + 0.62418624e^{-4i\omega} + 0.2432920962e^{2i\omega} + 0.35263374e^{4i\omega} \\ & + 0.1785179752(e^{i\omega} + e^{-3i\omega}) + 0.5053635204e^{3i\omega} - 0.704757e^{-6i\omega} \\ & + 0.630173943e^{-5i\omega} - 0.75784587748e^{-i\omega} + 0.3645086520399e^{-2i\omega},\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{21}^{(2)}(\omega) = & 0.176569773 + 0.30131206e^{-2i\omega} + 0.375198267e^{-4i\omega} - 0.515257857e^{-i\omega} \\ & + 0.524286047e^{2i\omega} + 0.26742134e^{4i\omega} + 0.128031385(e^{i\omega} + e^{-3i\omega}) + 0.323055559e^{3i\omega} \\ & + 0.0129836105e^{-6i\omega} - 0.589149204e^{-5i\omega},\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{22}^{(2)}(\omega) = & -0.2018916250 + 0.9405361667e^{-4i\omega} + 0.915484423e^{2i\omega} + 0.95094917e^{4i\omega} \\ & - 0.0718532141e^{-6i\omega} + 0.25388136(e^{i\omega} + e^{-3i\omega}) + 0.316328384e^{-5i\omega} - 0.829177467e^{3i\omega} \\ & + 0.9377471844e^{-i\omega} + 0.0101296080e^{-2i\omega}.\end{aligned}$$

4.2.3 Uniform Symmetry

In this subsection, we follow the work in (the scalar case) in the paper [3]. We derive the proposition in (the vector case) which states that the framelets which are obtained by type

I multiple bi-frame multiresolution algorithms have uniform symmetry.

The type I multiple frame multifilter banks corresponding to algorithms with more steps can be written in general forms as the following. Let $\mathbf{R}(\omega)$ be the matrix of the form:

$$\mathbf{R}(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{L}_1(e^{-i\omega}) & \mathbf{L}_2(e^{-i\omega}) \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \quad (4.52)$$

and $\tilde{\mathbf{R}}(\omega) = (\mathbf{R}(\omega)^{-1})^*$ is the matrix of the form:

$$\tilde{\mathbf{R}}(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\mathbf{L}_1(e^{i\omega}) & \mathbf{I}_2 & \mathbf{0}_2 \\ -\mathbf{L}_2(e^{i\omega}) & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \quad (4.53)$$

where $\mathbf{L}_1(e^{-i\omega})$ and $\mathbf{L}_2(e^{-i\omega})$ are Laurent matrix polynomials satisfying

$$\mathbf{L}_1(e^{i\omega}) = e^{-i\omega} \mathbf{L}_1(e^{-i\omega}), \quad \mathbf{L}_2(e^{i\omega}) = e^{-i\omega} \mathbf{L}_2(e^{-i\omega}), \quad (4.54)$$

then $\mathbf{L}_1(e^{-i\omega})$ and $\mathbf{L}_2(e^{-i\omega})$ are Laurent matrix polynomials of the form:

$$\mathbf{L}(e^{-i\omega}) = (1 + e^{i\omega})\mathbf{S}_1 + (e^{-i\omega} + e^{2i\omega})\mathbf{S}_2 + \dots + (e^{-(m-1)i\omega} + e^{mi\omega})\mathbf{S}_m, \quad (4.55)$$

where m is a positive integer and \mathbf{S}_s are real-valued 2×2 matrices, $s = 1, 2, \dots, m$. Then the type I multiple frame multifilter banks corresponding to algorithms with K ($K \geq 1$) steps can be written as the following

$$\begin{bmatrix} \mathbf{P}(\omega) \\ \mathbf{Q}^{(1)}(\omega) \\ \mathbf{Q}^{(2)}(\omega) \end{bmatrix} = \mathbf{B}_{K-1}(2\omega) \mathbf{B}_{K-2}(2\omega) \dots \mathbf{B}_1(2\omega) \mathbf{B}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega} \mathbf{I}_2 \end{bmatrix}, \quad (4.56)$$

$$\begin{bmatrix} \tilde{\mathbf{P}}(\omega) \\ \tilde{\mathbf{Q}}^{(1)}(\omega) \\ \tilde{\mathbf{Q}}^{(2)}(\omega) \end{bmatrix} = \frac{1}{2} \tilde{\mathbf{B}}_{K-1}(2\omega) \tilde{\mathbf{B}}_{K-2}(2\omega) \dots \tilde{\mathbf{B}}_1(2\omega) \tilde{\mathbf{B}}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega} \mathbf{I}_2 \end{bmatrix}, \quad (4.57)$$

where in 2-Step type I, we replace the 2×2 matrices \mathbf{B}_0 in (4.56) by \mathbf{C}_0 which is defined in (4.32) and we replace $\tilde{\mathbf{B}}_0$ in (4.57) by $\tilde{\mathbf{C}}_0$ in (4.33). In 3-Step type I, the matrices $\mathbf{B}_0(2\omega)$ and $\tilde{\mathbf{B}}_0(2\omega)$ are given in (4.48) and (4.49), each $\mathbf{B}_k(\omega)$, $1 \leq k \leq K-1$, is a matrix $\mathbf{R}(\omega)$ of the form (4.52) or $\tilde{\mathbf{R}}(\omega)$ of the form (4.53), where $\tilde{\mathbf{B}}_k(\omega) = (\mathbf{B}_k(\omega)^{-1})^*$.

The next proposition is derived from proposition 1 in the paper [3].

Proposition 1. Let $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ be the biorthogonal frame multi-filter banks defined by (4.56) and (4.57). Then

$$\begin{aligned}\mathbf{P}(-\omega) &= \mathbf{P}(\omega), \quad \mathbf{Q}^{(1)}(-\omega) = e^{i2\omega} \mathbf{Q}^{(1)}(\omega), \quad \mathbf{Q}^{(2)}(-\omega) = e^{i2\omega} \mathbf{Q}^{(2)}(\omega), \quad \tilde{\mathbf{P}}(-\omega) = \tilde{\mathbf{P}}(\omega), \\ \tilde{\mathbf{Q}}^{(1)}(-\omega) &= e^{i2\omega} \tilde{\mathbf{Q}}^{(1)}(\omega), \quad \tilde{\mathbf{Q}}^{(2)}(-\omega) = e^{i2\omega} \tilde{\mathbf{Q}}^{(2)}(\omega).\end{aligned}\tag{4.58}$$

Therefore, the associated multiscaling function $\Phi, \tilde{\Phi}$ and multiwavelets $\Psi^{(\ell)}, \tilde{\Psi}^{(\ell)}$, $\ell = 1, 2$, satisfy

$$\Phi(x) = \Phi(-x), \quad \Psi^{(\ell)}(x) = \Psi^{(\ell)}(1-x), \quad \tilde{\Phi}(x) = \tilde{\Phi}(-x), \quad \tilde{\Psi}^{(\ell)}(x) = \tilde{\Psi}^{(\ell)}(1-x).\tag{4.59}$$

Proof. From (4.52), we can easily verify that

$$\begin{aligned}\mathbf{B}_K(-\omega) &= \text{diag}(\mathbf{I}_2, e^{i\omega} \mathbf{I}_2, e^{i\omega} \mathbf{I}_2) \mathbf{B}_K(\omega) \text{diag}(\mathbf{I}_2, e^{-i\omega} \mathbf{I}_2, e^{-i\omega} \mathbf{I}_2), \\ \mathbf{B}_0(-\omega) &= \text{diag}(\mathbf{I}_2, e^{i\omega} \mathbf{I}_2, e^{i\omega} \mathbf{I}_2) \mathbf{B}_0(\omega) \text{diag}(\mathbf{I}_2, e^{-i\omega} \mathbf{I}_2, e^{-i\omega} \mathbf{I}_2),\end{aligned}\tag{4.60}$$

where $1 \leq k \leq K-1$. (4.60) implies

$$\begin{bmatrix} \mathbf{P}(-\omega) \\ \mathbf{Q}^{(1)}(-\omega) \\ \mathbf{Q}^{(2)}(-\omega) \end{bmatrix} = \text{diag}(\mathbf{I}_2, e^{i2\omega} \mathbf{I}_2, e^{i2\omega} \mathbf{I}_2) \begin{bmatrix} \mathbf{P}(\omega) \\ \mathbf{Q}^{(1)}(\omega) \\ \mathbf{Q}^{(2)}(\omega) \end{bmatrix}.$$

Since $\tilde{\mathbf{B}}_K(\omega)$ and $\tilde{\mathbf{B}}_0(\omega)$ satisfy (4.60), then we can obtain the symmetry of $\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}$.

From $\hat{\Phi}(\omega) = \prod_{j=1}^{\infty} \mathbf{P}(2^{-j}\omega) \hat{\Phi}(0)$ and $\mathbf{P}(-\omega) = \mathbf{P}(\omega)$, we have

$$\hat{\Phi}(-\omega) = \prod_{j=1}^{\infty} \mathbf{P}(-2^{-j}\omega) \hat{\Phi}(0) = \prod_{j=1}^{\infty} \mathbf{P}(2^{-j}\omega) \hat{\Phi}(0) = \hat{\Phi}(\omega).$$

Thus, $\Phi(-x) = \Phi(x)$.

From $\hat{\Psi}^{(\ell)}(\omega) = \mathbf{Q}^{(\ell)}(\frac{\omega}{2}) \hat{\Phi}(\frac{\omega}{2})$ and $\mathbf{Q}^{(\ell)}(-\omega) = e^{i2\omega} \mathbf{Q}^{(\ell)}(\omega)$, we have

$$\hat{\Psi}^{(\ell)}(-\omega) = \mathbf{Q}^{(\ell)}(-\frac{\omega}{2}) \hat{\Phi}(-\frac{\omega}{2}) = e^{i\omega} \mathbf{Q}^{(\ell)}(\frac{\omega}{2}) \hat{\Phi}(\frac{\omega}{2}) = e^{i\omega} \hat{\Psi}^{(\ell)}(\omega).$$

Thus, $\Psi^{(\ell)}(-x) = \Psi^{(\ell)}(x+1)$.

Similarly, we can proof the symmetry of $\tilde{\Phi}$ and $\tilde{\Psi}^{(\ell)}$.

From (4.53), we can easily verify that

$$\begin{aligned}\tilde{\mathbf{B}}_K(-\omega) &= \text{diag}(\mathbf{I}_2, e^{i\omega}\mathbf{I}_2, e^{i\omega}\mathbf{I}_2)\tilde{\mathbf{B}}_K(\omega)\text{diag}(\mathbf{I}_2, e^{-i\omega}\mathbf{I}_2, e^{-i\omega}\mathbf{I}_2), \\ \tilde{\mathbf{B}}_0(-\omega) &= \text{diag}(\mathbf{I}_2, e^{i\omega}\mathbf{I}_2, e^{i\omega}\mathbf{I}_2)\tilde{\mathbf{B}}_0(\omega)\text{diag}(\mathbf{I}_2, e^{-i\omega}\mathbf{I}_2, e^{-i\omega}\mathbf{I}_2),\end{aligned}\quad (4.61)$$

where $1 \leq k \leq K-1$. (4.61) implies

$$\begin{bmatrix} \tilde{\mathbf{P}}(-\omega) \\ \tilde{\mathbf{Q}}^{(1)}(-\omega) \\ \tilde{\mathbf{Q}}^{(2)}(-\omega) \end{bmatrix} = \text{diag}(\mathbf{I}_2, e^{i2\omega}\mathbf{I}_2, e^{i2\omega}\mathbf{I}_2) \begin{bmatrix} \tilde{\mathbf{P}}(\omega) \\ \tilde{\mathbf{Q}}^{(1)}(\omega) \\ \tilde{\mathbf{Q}}^{(2)}(\omega) \end{bmatrix}.$$

Since $\mathbf{B}_K(\omega)$ and $\mathbf{B}_0(\omega)$ satisfy (4.60), then we can obtain the symmetry of $\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}$.

From $\hat{\tilde{\mathbf{F}}}(\omega) = \prod_{j=1}^{\infty} \tilde{\mathbf{P}}(2^{-j}\omega)\hat{\tilde{\mathbf{F}}}(0)$ and $\tilde{\mathbf{P}}(-\omega) = \tilde{\mathbf{P}}(\omega)$, we have

$$\hat{\tilde{\mathbf{F}}}(-\omega) = \prod_{j=1}^{\infty} \tilde{\mathbf{P}}(-2^{-j}\omega)\hat{\tilde{\mathbf{F}}}(0) = \prod_{j=1}^{\infty} \tilde{\mathbf{P}}(2^{-j}\omega)\hat{\tilde{\mathbf{F}}}(0) = \hat{\tilde{\mathbf{F}}}(\omega).$$

Thus, $\tilde{\mathbf{F}}(-x) = \tilde{\mathbf{F}}(x)$.

From $\hat{\tilde{\Psi}}^{(\ell)}(\omega) = \tilde{\mathbf{Q}}^{(\ell)}(\frac{\omega}{2})\hat{\tilde{\mathbf{F}}}(\frac{\omega}{2})$ and $\tilde{\mathbf{Q}}^{(\ell)}(-\omega) = e^{i2\omega}\tilde{\mathbf{Q}}^{(\ell)}(\omega)$, we have

$$\hat{\tilde{\Psi}}^{(\ell)}(-\omega) = \tilde{\mathbf{Q}}^{(\ell)}(-\frac{\omega}{2})\hat{\tilde{\mathbf{F}}}(-\frac{\omega}{2}) = e^{i\omega}\tilde{\mathbf{Q}}^{(\ell)}(\frac{\omega}{2})\hat{\tilde{\mathbf{F}}}(\frac{\omega}{2}) = e^{i\omega}\hat{\tilde{\Psi}}^{(\ell)}(\omega).$$

Thus $\tilde{\Psi}^{(\ell)}(-x) = \tilde{\Psi}^{(\ell)}(x+1)$.

4.2.4 Multiple Bi-frames with Uniform Symmetry: Type II

Let $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ are a pair of biorthogonal frame multifilter banks. Let $\tilde{\mathbf{c}}_k$ be the lowpass output and $\mathbf{d}_k^{(1)}, \mathbf{d}_k^{(2)}$ be the highpass outputs of the input \mathbf{c}_k defined by (4.10) with analysis frame multifilter bank $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$. For initial data $\{\mathbf{c}_k\}$, let

$$\mathbf{v}_k = \mathbf{c}_{2k}, \mathbf{e}_k = \mathbf{c}_{2k+1}, k \in \mathbf{Z}, \quad (4.62)$$

and denote

$$\tilde{\mathbf{v}}_k = \tilde{\mathbf{c}}_k, \tilde{\mathbf{f}}_k = \mathbf{d}_k^{(1)}, \tilde{\mathbf{e}}_k = \mathbf{d}_k^{(2)}, k \in \mathbf{Z}, \quad (4.63)$$

where $\mathbf{v}_k, \mathbf{e}_k, \tilde{\mathbf{v}}_k, \tilde{\mathbf{f}}_k$ and $\tilde{\mathbf{e}}_k$ are 1×2 row-vectors

In the subsection (type I frame algorithms), we associate both the highpass outputs $\tilde{\mathbf{f}}_k$ and $\tilde{\mathbf{e}}_k$ with odd node $2k+1$. In this subsection (type II frame algorithms), we associate both the lowpass output $\tilde{\mathbf{v}}_k$ and the highpass output $\tilde{\mathbf{f}}_k$ with an even node $2k$ and $\tilde{\mathbf{e}}_k$ with an odd node $2k+1$. Thus, the decomposition and reconstruction algorithms are represented by templates.

4.2.5 2-Step Type II Multiple Bi-frame Algorithm

In this subsection, we will consider a 2-step type II frame multiresolution algorithm. By (4.64)-(4.65), we give the decomposition algorithm and it shown in Fig. 4.5. We obtain $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{f}}$ which are associated with odd nodes $2k + 1$ and given in (4.64). Then, with the obtained $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{e}}$, we replace \mathbf{e} by $\tilde{\mathbf{e}}$ given in (4.65).

2-step Type II Multiple Bi-frame Decomposition Algorithm:

$$\text{Step 1. } \begin{cases} \tilde{\mathbf{v}} = \{\mathbf{v} - (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{D}\} \mathbf{B}^{-1}; \\ \tilde{\mathbf{f}} = \mathbf{v} - (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{N}; \end{cases} \quad (4.64)$$

$$\text{Step 2. } \tilde{\mathbf{e}} = \mathbf{e} - (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1) \mathbf{U} - (\tilde{\mathbf{f}}_0 + \tilde{\mathbf{f}}_1) \mathbf{W}. \quad (4.65)$$

2-step Type II Multiple Bi-frame Reconstruction Algorithm:

$$\text{Step 1. } \mathbf{e} = \tilde{\mathbf{e}} + (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1) \mathbf{U} + (\tilde{\mathbf{f}}_0 + \tilde{\mathbf{f}}_1) \mathbf{W}; \quad (4.66)$$

$$\text{Step 2. } \mathbf{v} = \{\tilde{\mathbf{v}} \mathbf{B} + (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{D}\} \mathbf{T} + \{\tilde{\mathbf{f}} + (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{N}\} (\mathbf{I}_2 - \mathbf{T}), \quad (4.67)$$

where \mathbf{B} , \mathbf{T} , \mathbf{D} , \mathbf{U} , \mathbf{W} , and \mathbf{N} are 2×2 matrices and the entries of these matrices are some constants in \mathbb{R} . Let

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}, \quad \text{and}$$

$$\mathbf{N} = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix},$$

where the entries of these square matrices are some constants in \mathbb{R} .

The reconstruction algorithm is shown in Fig.4.6. By (4.66)-(4.67), we give the multiresolution reconstruction algorithm, where the matrices are the same 2×2 matrices in the decomposition algorithm in \mathbb{R} .

First in Step 1, we update $\tilde{\mathbf{e}}$ of the highpass output associated with odd nodes by \mathbf{e} given in (4.66). This step recovers original data \mathbf{c}_{2k+1} associated with odd nodes $2k + 1$. Then, in Step 2 with the obtained \mathbf{e} in Step 1, we obtain \mathbf{v} given by formula (4.67). This step recovers original data \mathbf{c}_{2k} associated with even nodes $2k$. So the inputs here are $\tilde{\mathbf{v}}$, $\tilde{\mathbf{f}}$, $\tilde{\mathbf{e}}$ and the outputs are \mathbf{e} , \mathbf{v} .

Now, let us obtain the multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ corresponding to this 2-step frame algorithm.

From (4.63) and (4.64) we have

$$\tilde{\mathbf{c}}_n = \tilde{\mathbf{v}}_n = \{\mathbf{v}_n - (\mathbf{e}_{n-1} + \mathbf{e}_n) \mathbf{D}\} \mathbf{B}^{-1} = \{\mathbf{c}_{2n} - (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+1}) \mathbf{D}\} \mathbf{B}^{-1}. \quad (4.68)$$

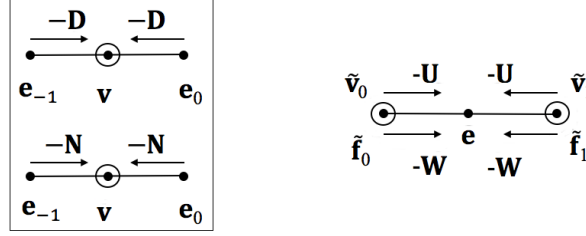


Figure 4.5: Left: Decomposition Step 1; Right: Decomposition Step 2.



Figure 4.6: Left: Reconstruction Step 1; Right: Reconstruction Step 2.

From (4.63) and (4.64) we have

$$\mathbf{d}_n^{(1)} = \tilde{\mathbf{f}}_n = \mathbf{v}_n - (\mathbf{e}_{n-1} + \mathbf{e}_n) \mathbf{N} = \mathbf{c}_{2n} - (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+1}) \mathbf{N}. \quad (4.69)$$

Similarly, from (4.63) and (4.65) we have

$$\begin{aligned} \mathbf{d}_n^{(2)} &= \tilde{\mathbf{e}}_n = \mathbf{e}_n - (\tilde{\mathbf{v}}_n + \tilde{\mathbf{v}}_{n+1}) \mathbf{U} - (\tilde{\mathbf{f}}_n + \tilde{\mathbf{f}}_{n+1}) \mathbf{W} \\ &= \mathbf{c}_{2n+1} - \left(\{ \mathbf{c}_{2n} - (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+1}) \mathbf{D} \} \mathbf{B}^{-1} + \{ \mathbf{c}_{2n+2} - (\mathbf{c}_{2n+1} + \mathbf{c}_{2n+3}) \mathbf{D} \} \mathbf{B}^{-1} \right) \mathbf{U} \\ &\quad - (\mathbf{c}_{2n} - (\mathbf{c}_{2n-1} + \mathbf{c}_{2n+1}) \mathbf{N} + \mathbf{c}_{2n+2} - (\mathbf{c}_{2n+1} + \mathbf{c}_{2n+3}) \mathbf{N}) \mathbf{W}. \end{aligned} \quad (4.70)$$

We get the nonzero coefficients \mathbf{P}_k , $\mathbf{Q}_k^{(1)}$, $\mathbf{Q}_k^{(2)}$ of $\{\mathbf{P}(\omega), \mathbf{Q}^{(1)}(\omega), \mathbf{Q}^{(2)}(\omega)\}$ by comparing (4.68), (4.69) and (4.70) with (4.10):

$$\begin{aligned} \mathbf{P}_0 &= 2(\mathbf{B}^{-1})^T, \quad \mathbf{P}_{-1} = \mathbf{P}_1 = -2(\mathbf{B}^{-1})^T \mathbf{D}^T; \\ \mathbf{Q}_0^{(1)} &= 2\mathbf{I}_2, \quad \mathbf{Q}_1^{(1)} = \mathbf{Q}_{-1}^{(1)} = -2\mathbf{N}^T; \\ \mathbf{Q}_0^{(2)} &= \mathbf{Q}_2^{(2)} = -2\mathbf{U}^T (\mathbf{B}^{-1})^T - 2\mathbf{W}^T, \quad \mathbf{Q}_1^{(2)} = 2\mathbf{I}_2 + 4\mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T + 4\mathbf{W}^T \mathbf{N}^T, \\ \mathbf{Q}_{-1}^{(2)} &= \mathbf{Q}_3^{(2)} = 2\mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T + 2\mathbf{W}^T \mathbf{N}^T. \end{aligned}$$

Therefore, the analysis multifilter bank $\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}$ is

$$\mathbf{P}(\omega) = \frac{1}{2}(2(\mathbf{B}^{-1})^T - 2(e^{i\omega} + e^{-i\omega})(\mathbf{B}^{-1})^T \mathbf{D}^T); \quad (4.71)$$

$$\mathbf{Q}^{(1)}(\omega) = \frac{1}{2}(2\mathbf{I}_2 - 2(e^{i\omega} + e^{-i\omega})\mathbf{N}^T); \quad (4.72)$$

$$\begin{aligned} \mathbf{Q}^{(2)}(\omega) = & \frac{1}{2}(e^{-i\omega}(2\mathbf{I}_2 + 4\mathbf{U}^T(\mathbf{B}^{-1})^T \mathbf{D}^T + 4\mathbf{W}^T \mathbf{N}^T) + (1 + e^{-2i\omega})(-2\mathbf{U}^T(\mathbf{B}^{-1})^T \\ & - 2\mathbf{W}^T) + (e^{i\omega} + e^{-3i\omega})(2\mathbf{U}^T(\mathbf{B}^{-1})^T \mathbf{D}^T + 2\mathbf{W}^T \mathbf{N}^T)). \end{aligned} \quad (4.73)$$

We will obtain the synthesis multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ from (4.62) and (4.66) we have

$$\begin{aligned} \mathbf{c}_{2k+1} = \mathbf{e}_k &= \tilde{\mathbf{e}}_k + (\tilde{\mathbf{v}}_k + \tilde{\mathbf{v}}_{k+1})\mathbf{U} + (\tilde{\mathbf{f}}_k + \tilde{\mathbf{f}}_{k+1})\mathbf{W}, \\ &= \mathbf{d}_k^{(2)} + (\tilde{\mathbf{c}}_k + \tilde{\mathbf{c}}_{k+1})\mathbf{U} + (\mathbf{d}_k^{(1)} + \mathbf{d}_{k+1}^{(1)})\mathbf{W}. \end{aligned} \quad (4.74)$$

(4.62) and (4.67) gives us

$$\begin{aligned} \mathbf{c}_{2k} = \mathbf{v}_k &= \{\tilde{\mathbf{v}}_k \mathbf{B} + (\mathbf{e}_{k-1} + \mathbf{e}_k)\mathbf{D}\}\mathbf{T} + \{\tilde{\mathbf{f}}_k + (\mathbf{e}_{k-1} + \mathbf{e}_k)\mathbf{N}\}(\mathbf{I}_2 - \mathbf{T}) \\ &= \{\tilde{\mathbf{c}}_k \mathbf{B} + (\mathbf{d}_{k-1}^{(2)} + (\tilde{\mathbf{c}}_{k-1} + \tilde{\mathbf{c}}_k)\mathbf{U} + (\mathbf{d}_{k-1}^{(1)} + \mathbf{d}_k^{(1)})\mathbf{W} \\ &\quad + \mathbf{d}_k^{(2)} + (\tilde{\mathbf{c}}_k + \tilde{\mathbf{c}}_{k+1})\mathbf{U} + (\mathbf{d}_k^{(1)} + \mathbf{d}_{k+1}^{(1)})\mathbf{W})\mathbf{D}\}\mathbf{T} + \{\mathbf{d}_k^{(1)} \\ &\quad + (\mathbf{d}_{k-1}^{(2)} + (\tilde{\mathbf{c}}_{k-1} + \tilde{\mathbf{c}}_k)\mathbf{U} + (\mathbf{d}_{k-1}^{(1)} + \mathbf{d}_k^{(1)})\mathbf{W} + \mathbf{d}_k^{(2)} + (\tilde{\mathbf{c}}_k \\ &\quad + \tilde{\mathbf{c}}_{k+1})\mathbf{U} + (\mathbf{d}_k^{(1)} + \mathbf{d}_{k+1}^{(1)})\mathbf{W})\mathbf{N}\}\mathbf{T}\}(\mathbf{I}_2 - \mathbf{T}). \end{aligned} \quad (4.75)$$

By comparing (4.74) and (4.11) for odd $2k + 1$, we have that the nonzero coefficients $\mathbf{P}_{2k+1}, \mathbf{Q}_{2k+1}^{(1)}, \mathbf{Q}_{2k+1}^{(2)}$ with odd indices $2k + 1$ are

$$\tilde{\mathbf{P}}_1 = \tilde{\mathbf{P}}_{-1} = \mathbf{U}, \quad \tilde{\mathbf{Q}}_1^{(1)} = \tilde{\mathbf{Q}}_{-1}^{(1)} = \mathbf{W}, \quad \tilde{\mathbf{Q}}_1^{(2)} = \mathbf{I}_2,$$

where the other coefficients are zero.

Now, we compare (4.75) with (4.11) for even $2k$, we have that the nonzero coefficients $\mathbf{P}_{2k}, \mathbf{Q}_{2k}^{(1)}, \mathbf{Q}_{2k}^{(2)}$ with even indices $2k$ are

$$\begin{aligned} \tilde{\mathbf{P}}_0 &= \mathbf{B}\mathbf{T} + 2\mathbf{U}\mathbf{D}\mathbf{T} + 2\mathbf{U}\mathbf{N}(\mathbf{I}_2 - \mathbf{T}), \quad \tilde{\mathbf{P}}_2 = \tilde{\mathbf{P}}_{-2} = \mathbf{U}\mathbf{D}\mathbf{T} + \mathbf{U}\mathbf{N}(\mathbf{I}_2 - \mathbf{T}); \\ \tilde{\mathbf{Q}}_0^{(1)} &= 2\mathbf{W}\mathbf{D}\mathbf{T} + 2\mathbf{W}\mathbf{N}(\mathbf{I}_2 - \mathbf{T}) + (\mathbf{I}_2 - \mathbf{T}), \quad \tilde{\mathbf{Q}}_2^{(1)} = \tilde{\mathbf{Q}}_{-2}^{(1)} = \mathbf{W}\mathbf{D}\mathbf{T} + \mathbf{W}\mathbf{N}(\mathbf{I}_2 - \mathbf{T}); \\ \tilde{\mathbf{Q}}_0^{(2)} &= \tilde{\mathbf{Q}}_2^{(2)} = \mathbf{D}\mathbf{T} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T}). \end{aligned}$$

Hence the synthesis multifilter bank is

$$\begin{aligned} \tilde{\mathbf{P}}(\omega) = & \frac{1}{2}(\mathbf{BT} + 2\mathbf{UDT} + 2\mathbf{UN}(\mathbf{I}_2 - \mathbf{T}) + (e^{i\omega} + e^{-i\omega})\mathbf{U} + (e^{2i\omega} + e^{-2i\omega})(\mathbf{UDT} \\ & + \mathbf{UN}(\mathbf{I}_2 - \mathbf{T}))); \end{aligned} \quad (4.76)$$

$$\begin{aligned} \tilde{\mathbf{Q}}^{(1)}(\omega) = & \frac{1}{2}(2\mathbf{WDT} + 2\mathbf{WN}(\mathbf{I}_2 - \mathbf{T}) + (\mathbf{I}_2 - \mathbf{T}) + (e^{i\omega} + e^{-i\omega})\mathbf{W} + (e^{2i\omega} \\ & + e^{-2i\omega})(\mathbf{WDT} + \mathbf{WN}(\mathbf{I}_2 - \mathbf{T}))); \end{aligned} \quad (4.77)$$

$$\tilde{\mathbf{Q}}^{(2)}(\omega) = \frac{1}{2}((1 + e^{-2i\omega})(\mathbf{DT} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T})) + e^{-i\omega}\mathbf{I}_2). \quad (4.78)$$

Denote

$$\begin{cases} \mathbf{D}_1(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ -(1 + e^{-i\omega})\mathbf{U}^T & -(1 + e^{-i\omega})\mathbf{W}^T & \mathbf{I}_2 \end{bmatrix}, \\ \mathbf{D}_0(\omega) = \begin{bmatrix} (\mathbf{B}^{-1})^T & -(1 + e^{i\omega})(\mathbf{B}^{-1})^T\mathbf{D}^T \\ \mathbf{I}_2 & -(1 + e^{i\omega})\mathbf{N}^T \\ \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \end{cases} \quad (4.79)$$

and

$$\begin{cases} \tilde{\mathbf{D}}_1(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & (1 + e^{i\omega})\mathbf{U} \\ \mathbf{0}_2 & \mathbf{I}_2 & (1 + e^{i\omega})\mathbf{W} \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \\ \tilde{\mathbf{D}}_0(\omega) = \begin{bmatrix} \mathbf{BT} & \mathbf{0}_2 \\ (\mathbf{I}_2 - \mathbf{T}) & \mathbf{0}_2 \\ (1 + e^{-i\omega})(\mathbf{DT} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T})) & \mathbf{I}_2 \end{bmatrix}. \end{cases} \quad (4.80)$$

Then $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ can be written as

$$\begin{bmatrix} \mathbf{P}(\omega) \\ \mathbf{Q}^{(1)}(\omega) \\ \mathbf{Q}^{(2)}(\omega) \end{bmatrix} = \mathbf{D}_1(2\omega)\mathbf{D}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}, \quad (4.81)$$

and

$$\begin{bmatrix} \tilde{\mathbf{P}}(\omega) \\ \tilde{\mathbf{Q}}^{(1)}(\omega) \\ \tilde{\mathbf{Q}}^{(2)}(\omega) \end{bmatrix} = \frac{1}{2}\tilde{\mathbf{D}}_1(2\omega)\tilde{\mathbf{D}}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}, \quad (4.82)$$

First, we solve the system of equations for a sum rule order one of $\tilde{\mathbf{P}}$, a sum rule order one of \mathbf{P} , and for a vanishing moment order one of both $\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}$, we have 12 free parameters which are $b_{22}, d_{12}, d_{22}, n_{12}, n_{22}, t_{11}, t_{12}, t_{22}, u_{22}, w_{11}, w_{12}, w_{22}$.

$$b_{11} = 2, b_{12} = -\frac{2(d_{12}t_{22} - n_{12}t_{22} + n_{12})}{t_{22}}, b_{21} = 0, d_{11} = -\frac{1}{2}, d_{21} = 0,$$

$$n_{11} = \frac{1}{2}, n_{21} = 0, t_{21} = 0, u_{11} = \frac{1}{2}, u_{12} = 0, u_{21} = 0, w_{21} = 0.$$

If we choose

$$[b_{22}, d_{12}, d_{22}, n_{12}, n_{22}, t_{11}, t_{12}, t_{22}, u_{22}, w_{11}, w_{12}, w_{22}] = \left[\frac{439}{211}, \frac{121}{577}, \frac{88}{315}, \frac{169}{617}, \frac{211}{334}, \frac{51}{1811}, \right. \\ \left. \frac{69}{1285}, \frac{-101}{725}, \frac{20}{301}, \frac{-179}{703}, \frac{383}{606} \right].$$

Using the smoothness formula, we can obtain $\tilde{\Phi} \in W^{1.9994}$.

By solving the system of equations for a sum rule order four of $\tilde{\mathbf{P}}$, a sum rule order two of \mathbf{P} , and for a vanishing moment order one of both $\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}$, we have 10 free parameters which are $b_{22}, d_{12}, d_{22}, n_{12}, n_{22}, t_{22}, u_{22}, w_{11}, w_{12}, w_{22}$.

$$b_{11} = 2, b_{12} = -\frac{2(d_{12}t_{22} - n_{12}t_{22} + n_{12})}{t_{22}}, b_{21} = 0, d_{11} = -\frac{1}{2}, d_{21} = 0, g = 0,$$

$$h = -\frac{1}{3}, n_{11} = \frac{1}{2}, n_{21} = 0, t_{11} = \frac{1}{4}, t_{12} = d_{12}t_{22} - n_{12}t_{22} + n_{12}, t_{21} = 0,$$

$$td1 = 0, td2 = 0, u_{11} = \frac{1}{2}, u_{12} = 0, u_{21} = 0, w_{21} = 0, x = 0, z = 0.$$

If we choose

$$[b_{22}, d_{12}, d_{22}, n_{12}, n_{22}, t_{22}, u_{22}, w_{11}, w_{12}, w_{22}] = \left[\frac{929}{191}, \frac{203}{237}, \frac{-9}{410}, \frac{819}{1069}, \frac{41}{460}, \frac{-8}{185}, \frac{101}{1060}, \frac{123}{220}, \frac{255}{287}, \right. \\ \left. \frac{257}{289} \right].$$

By using the smoothness formula, we can obtain $\tilde{\Phi} \in W^{2.6540}$ and $\Phi \in W^{1.5}$.

The lowpass analysis mask \mathbf{P} and the lowpass synthesis mask $\tilde{\mathbf{P}}$ will be as follows:

$$\mathbf{P}(\omega) = \begin{bmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{bmatrix}$$

where

$$P_{11}(\omega) = -0.25(e^{i\omega} + e^{-i\omega}) + 0.5,$$

$$P_{12}(\omega) = 0,$$

$$P_{21}(\omega) = -1.9880843554(e^{i\omega} + e^{-i\omega}) - 3.6239638537366$$

$$P_{22}(\omega) = -0.0045131140224(e^{i\omega} + e^{-i\omega}) + 0.2055974165769.$$

and

$$\tilde{\mathbf{P}}(\omega) = \begin{bmatrix} \tilde{P}_{11}(\omega) & \tilde{P}_{12}(\omega) \\ \tilde{P}_{21}(\omega) & \tilde{P}_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} \tilde{P}_{11}(\omega) &= 0.0625(e^{2i\omega} + e^{-2i\omega}) + 0.375 + 0.25(e^{i\omega} + e^{-i\omega}), \\ \tilde{P}_{12}(\omega) &= 0, \\ \tilde{P}_{21}(\omega) &= 0, \\ \tilde{P}_{22}(\omega) &= -0.0044751559168(e^{2i\omega} + e^{-2i\omega}) - 0.1141151625482 - 0.0476415094339(e^{i\omega} + e^{-i\omega}). \end{aligned}$$

Thus, the resulting Φ is the linear B -spline supported on $[-1, 1]$. The highpass analysis mask \mathbf{Q} and the highpass synthesis mask $\tilde{\mathbf{Q}}$ will be as follows:

$$\mathbf{Q}_1^{(1)}(\omega) = \begin{bmatrix} Q_{11}^{(1)}(\omega) & Q_{12}^{(1)}(\omega) \\ Q_{21}^{(1)}(\omega) & Q_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned} Q_{11}^{(1)}(\omega) &= -0.5(e^{i\omega} + e^{-i\omega}) + 1, \\ Q_{12}^{(1)}(\omega) &= 0, \\ Q_{21}^{(1)}(\omega) &= -0.7661365762394(e^{i\omega} + e^{-i\omega}), \\ Q_{22}^{(1)}(\omega) &= -0.0021739130434(e^{i\omega} + e^{-i\omega}) + 1. \end{aligned}$$

$$\mathbf{Q}_2^{(2)}(\omega) = \begin{bmatrix} Q_{11}^{(2)}(\omega) & Q_{12}^{(2)}(\omega) \\ Q_{21}^{(2)}(\omega) & Q_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned} Q_{11}^{(2)}(\omega) &= -0.4045454545454(e^{i\omega} + e^{-3i\omega}) + 0.3090909090909(1 + e^{-2i\omega}) + 0.1909090909090e^{-i\omega}, \\ Q_{12}^{(2)}(\omega) &= 0, \\ Q_{21}^{(2)}(\omega) &= 0.9355338746537(e^{i\omega} + e^{-3i\omega}) - 1.2338039584125(1 + e^{-2i\omega}) \\ &\quad + 1.8710677493075e^{-i\omega}, \\ Q_{22}^{(2)}(\omega) &= 0.079622569712(e^{i\omega} + e^{-3i\omega}) - 0.8689117996373(1 + e^{-2i\omega}) + 1.159245139424e^{-i\omega}. \end{aligned}$$

and

$$\tilde{\mathbf{Q}}_1^{(1)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(1)}(\omega) & \tilde{Q}_{12}^{(1)}(\omega) \\ \tilde{Q}_{21}^{(1)}(\omega) & \tilde{Q}_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}\tilde{Q}_{11}^{(1)}(\omega) &= -0.0698863636363(e^{2i\omega} + e^{-2i\omega}) - 0.2795454545454(e^{i\omega} + e^{-i\omega}) + 0.2352272727272, \\ \tilde{Q}_{12}^{(1)}(\omega) &= 0.041730246121(e^{2i\omega} + e^{-2i\omega}) + 0.4442508710801(e^{i\omega} + e^{-i\omega}) + -0.2976531254304, \\ \tilde{Q}_{21}^{(1)}(\omega) &= 0, \\ \tilde{Q}_{22}^{(1)}(\omega) &= 0.041730246121(e^{2i\omega} + e^{-2i\omega}) + 0.4442508710801(e^{i\omega} + e^{-i\omega}) + 0.6050821138644.\end{aligned}$$

$$\tilde{\mathbf{Q}}_2^{(2)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(2)}(\omega) & \tilde{Q}_{12}^{(2)}(\omega) \\ \tilde{Q}_{21}^{(2)}(\omega) & \tilde{Q}_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}\tilde{Q}_{11}^{(2)}(\omega) &= 0.125(1 + e^{-2i\omega}) + 0.5e^{-i\omega}, \\ \tilde{Q}_{12}^{(2)}(\omega) &= 0, \\ \tilde{Q}_{21}^{(2)}(\omega) &= 0, \\ \tilde{Q}_{22}^{(2)}(\omega) &= 0.0469669828895(1 + e^{-2i\omega}) + 0.5e^{-i\omega}.\end{aligned}$$

4.2.6 3-Step Type II Multiple Bi-frame Algorithm

In this subsection, let us consider a 3-step type II mutiple bi-frame multiresolution algorithm. In (4.83)-(4.85), we give the decomposition algorithm and it shown in Fig. 4.7. \mathbf{v}'' and \mathbf{f}'' both are associated with even nodes $2k$. The one highpass output $\tilde{\mathbf{e}}$ is associated with odd nodes $2k + 1$.

3-step Type II Multiple Bi-frame Decompostion Algorithm:

$$\text{Step 1. } \begin{cases} \mathbf{v}'' = \{\mathbf{v} - (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{D}\} \mathbf{B}^{-1}, \\ \mathbf{f}'' = \mathbf{v} - (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{N}; \end{cases} \quad (4.83)$$

$$\text{Step 2. } \tilde{\mathbf{e}} = \mathbf{e} - (\mathbf{v}_0'' + \mathbf{v}_1'') \mathbf{U} - (\mathbf{f}_0'' + \mathbf{f}_1'') \mathbf{W}; \quad (4.84)$$

$$\text{Step 3. } \begin{cases} \tilde{\mathbf{v}} = \mathbf{v}'' - (\tilde{\mathbf{e}}_{-1} + \tilde{\mathbf{e}}_0) \mathbf{D}_1, \\ \tilde{\mathbf{f}} = \mathbf{f}'' - (\tilde{\mathbf{e}}_{-1} + \tilde{\mathbf{e}}_0) \mathbf{N}_1. \end{cases} \quad (4.85)$$

3-step Type II Multiple Bi-frame Reconstruction Algorithm:

$$\text{Step 1. } \begin{cases} \mathbf{v}'' = \tilde{\mathbf{v}} + (\tilde{\mathbf{e}}_{-1} + \tilde{\mathbf{e}}_0) \mathbf{D}_1, \\ \mathbf{f}'' = \tilde{\mathbf{f}} + (\tilde{\mathbf{e}}_{-1} + \tilde{\mathbf{e}}_0) \mathbf{N}_1; \end{cases} \quad (4.86)$$

$$\text{Step 2. } \mathbf{e} = \tilde{\mathbf{e}} + (\mathbf{v}_0'' + \mathbf{v}_1'') \mathbf{U} + (\mathbf{f}_0'' + \mathbf{f}_1'') \mathbf{W}; \quad (4.87)$$

$$\text{Step 3. } \mathbf{v} = \{\mathbf{v}'' \mathbf{B} + (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{D}\} \mathbf{T} + \{\mathbf{f}'' + (\mathbf{e}_{-1} + \mathbf{e}_0) \mathbf{N}\} (\mathbf{I}_2 - \mathbf{T}), \quad (4.88)$$

where \mathbf{B} , \mathbf{D} , \mathbf{U} , \mathbf{W} , \mathbf{D}_1 , \mathbf{N}_1 , and \mathbf{T} are 2×2 matrices and the entries of these matrices are some constants in \mathbb{R} .

The 3-step decomposition algorithm is given in (4.83)-(4.85) and shown in Fig. 4.7. In Step 1, we obtain \mathbf{v}'' and \mathbf{f}'' associated with even nodes $2k$ by the formulas in (4.83). Then in Step 2, with the obtained \mathbf{v}'' , \mathbf{f}'' , we obtain one highpass output $\tilde{\mathbf{e}}$ that is associated with odd nodes $2k + 1$ and given by (4.84). Finally in Step 3, with the obtained $\tilde{\mathbf{e}}$, we replace \mathbf{v}'' and \mathbf{f}'' by $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{f}}$ respectively which is given in (4.85). This step gives the lowpass output $\tilde{\mathbf{v}}$ and the other highpass output $\tilde{\mathbf{f}}$ which are associated with even nodes $2k$.

The reconstruction algorithm is given in (4.86)-(4.88) and shown in Fig. 4.8, where the matrices are the same as in the decomposition algorithm. In Step 1, we obtain \mathbf{v}'' and \mathbf{f}'' associated with even nodes $2k$ by (4.86). In Step 2, with the obtained \mathbf{v}'' and \mathbf{f}'' , we replace $\tilde{\mathbf{e}}$ of the highpass output associated with odd nodes $2k + 1$ by \mathbf{e} given in (4.87). This step recovers original data \mathbf{c}_{2k+1} associated with odd nodes $2k + 1$. Finally in Step 3, we obtain \mathbf{v} from the formula (4.88). In this step, we recover the original data \mathbf{c}_{2k} which is associated with even nodes.

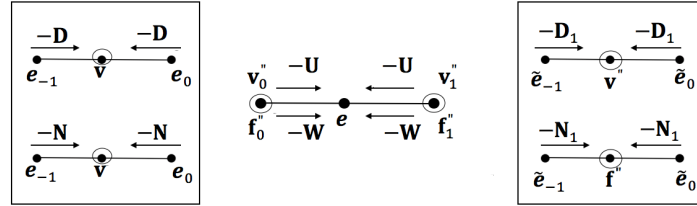


Figure 4.7: Left: decomposition Step 1; middle: decomposition Step 2; right: decomposition Step 3.

Therefore, the analysis multifilter bank

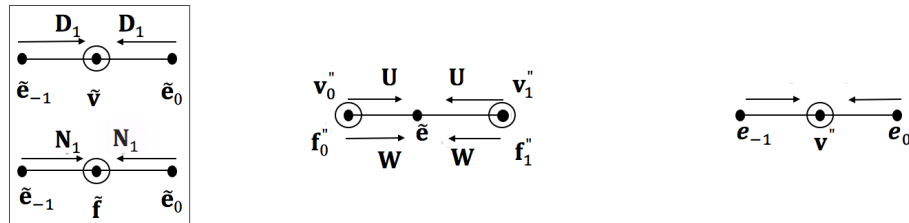


Figure 4.8: Left: Reconstruction Step 1; middle: Reconstruction Step 2; right: Reconstruction Step 3.

$$\begin{aligned}
\mathbf{P}(\omega) = & \frac{1}{2}(2(\mathbf{B}^{-1})^T + 4\mathbf{D}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T + 4\mathbf{D}_1^T \mathbf{W}^T \\
& + (e^{i\omega} + e^{-i\omega})(-2(\mathbf{B}^{-1})^T \mathbf{D}^T - 6\mathbf{D}_1^T \mathbf{W}^T \mathbf{N}^T - 2\mathbf{D}_1^T - 6\mathbf{D}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T) \\
& + (e^{2i\omega} + e^{-2i\omega})(2\mathbf{D}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T + 2\mathbf{D}_1^T \mathbf{W}^T) \\
& + (e^{3i\omega} + e^{-3i\omega})(-2\mathbf{D}_1^T \mathbf{W}^T \mathbf{N}^T - 2\mathbf{D}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T)); \tag{4.89}
\end{aligned}$$

$$\begin{aligned}
\mathbf{Q}^{(1)}(\omega) = & \frac{1}{2}(4\mathbf{N}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T + 4\mathbf{N}_1^T \mathbf{W}^T + 2\mathbf{I}_2 + (e^{i\omega} + e^{-i\omega})(-6\mathbf{N}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T \\
& - 6\mathbf{N}_1^T \mathbf{W}^T \mathbf{N}^T - 2\mathbf{N}_1^T) + (e^{2i\omega} + e^{-2i\omega})(2\mathbf{N}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T + 2\mathbf{N}_1^T \mathbf{W}^T) \\
& + (e^{3i\omega} + e^{-3i\omega})(-6\mathbf{N}_1^T \mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T - 6\mathbf{N}_1^T \mathbf{W}^T \mathbf{N}^T)); \tag{4.90}
\end{aligned}$$

$$\begin{aligned}
\mathbf{Q}^{(2)}(\omega) = & \frac{1}{2}((1 + e^{-2i\omega})(-2\mathbf{U}^T (\mathbf{B}^{-1})^T - 2\mathbf{W}^T) + (e^{i\omega} + e^{-3i\omega})(2\mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T \\
& + 2\mathbf{W}^T \mathbf{N}^T) + e^{-i\omega}(4\mathbf{U}^T (\mathbf{B}^{-1})^T \mathbf{D}^T + 4\mathbf{W}^T \mathbf{N}^T + 2\mathbf{I}_2)). \tag{4.91}
\end{aligned}$$

The synthesis multifilter bank is

$$\begin{aligned}
\tilde{\mathbf{P}}(\omega) = & \frac{1}{2}(\mathbf{B}\mathbf{T} + 2\mathbf{U}\mathbf{D}\mathbf{T} + 2\mathbf{U}\mathbf{N}(\mathbf{I}_2 - \mathbf{T}) + (e^{i\omega} + e^{-i\omega})\mathbf{U} + (e^{2i\omega} + e^{-2i\omega})(\mathbf{U}\mathbf{D}\mathbf{T} \\
& + \mathbf{U}\mathbf{N}(\mathbf{I}_2 - \mathbf{T}))); \tag{4.92}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{Q}}^{(1)}(\omega) = & \frac{1}{2}((\mathbf{I}_2 - \mathbf{T}) + 2\mathbf{W}\mathbf{D}\mathbf{T} + \mathbf{W}\mathbf{N}(\mathbf{I}_2 - \mathbf{T}) + (e^{i\omega} + e^{-i\omega})\mathbf{W} + (e^{2i\omega} + e^{-2i\omega}) \\
& \mathbf{W}\mathbf{D}\mathbf{T} + \mathbf{W}\mathbf{N}(\mathbf{I}_2 - \mathbf{T})); \tag{4.93}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{Q}}^{(2)}(\omega) = & \frac{1}{2}(\mathbf{D}_1\mathbf{B}\mathbf{T} + \mathbf{N}_1(\mathbf{I}_2 - \mathbf{T}) + 3\mathbf{D}_1\mathbf{U} + \mathbf{N}_1\mathbf{W}\mathbf{D}\mathbf{T} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T}) + e^{-i\omega}(\mathbf{I}_2 \\
& + 2\mathbf{D}_1\mathbf{U} + 2\mathbf{N}_1\mathbf{W}) + e^{-2i\omega}(\mathbf{D}_1\mathbf{B}\mathbf{T} + \mathbf{N}_1(\mathbf{I}_2 - \mathbf{T}) + 3\mathbf{D}_1\mathbf{U} + 3\mathbf{N}_1\mathbf{W} \\
& + \mathbf{I}_2\mathbf{D}\mathbf{T} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T})) + (e^{2i\omega} + e^{-4i\omega})(\mathbf{D}_1\mathbf{U} + \mathbf{N}_1\mathbf{W}\mathbf{D}\mathbf{T} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T})) \\
& + (e^{i\omega} + e^{-3i\omega})(\mathbf{D}_1\mathbf{U} + \mathbf{N}_1\mathbf{W})). \tag{4.94}
\end{aligned}$$

Denote

$$\left\{ \begin{aligned} \mathbf{D}_2(\omega) &= \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & -(1 + e^{i\omega})\mathbf{D}_1^T \\ \mathbf{0}_2 & \mathbf{I}_2 & -(1 + e^{i\omega})\mathbf{N}_1^T \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \\ \mathbf{D}_1(\omega) &= \begin{bmatrix} & \mathbf{I}_2 & & \mathbf{0}_2 & \mathbf{0}_2 \\ & \mathbf{0}_2 & & \mathbf{I}_2 & \mathbf{0}_2 \\ -(1 + e^{-i\omega})\mathbf{U}^T & & -(1 + e^{-i\omega})\mathbf{W}^T & & \mathbf{I}_2 \end{bmatrix}, \\ \mathbf{D}_0(\omega) &= \begin{bmatrix} (\mathbf{B}^{-1})^T & -(1 + e^{i\omega})(\mathbf{B}^{-1})^T \mathbf{D}^T \\ \mathbf{I}_2 & -(1 + e^{i\omega})\mathbf{N}^T \\ \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \end{aligned} \right. \tag{4.95}$$

and

$$\left\{ \begin{array}{l} \tilde{\mathbf{D}}_2(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ (1 + e^{-i\omega})\mathbf{D}_1 & (1 + e^{-i\omega})\mathbf{N}_1 & \mathbf{I}_2 \end{bmatrix}, \\ \tilde{\mathbf{D}}_1(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & (1 + e^{i\omega})\mathbf{U} \\ \mathbf{0}_2 & \mathbf{I}_2 & (1 + e^{i\omega})\mathbf{W} \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \\ \tilde{\mathbf{D}}_0(\omega) = \begin{bmatrix} \mathbf{B}\mathbf{T} & \mathbf{0}_2 \\ (\mathbf{I}_2 - \mathbf{T}) & \mathbf{0}_2 \\ (1 + e^{-i\omega})(\mathbf{D}\mathbf{T} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T})) & \mathbf{I}_2 \end{bmatrix}. \end{array} \right. \quad (4.96)$$

Then $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ can be written as

$$\begin{bmatrix} \mathbf{P}(\omega) \\ \mathbf{Q}^{(1)}(\omega) \\ \mathbf{Q}^{(2)}(\omega) \end{bmatrix} = \mathbf{D}_2(2\omega)\mathbf{D}_1(2\omega)\mathbf{D}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}, \quad (4.97)$$

and

$$\begin{bmatrix} \tilde{\mathbf{P}}(\omega) \\ \tilde{\mathbf{Q}}^{(1)}(\omega) \\ \tilde{\mathbf{Q}}^{(2)}(\omega) \end{bmatrix} = \frac{1}{2}\tilde{\mathbf{D}}_2(2\omega)\tilde{\mathbf{D}}_1(2\omega)\tilde{\mathbf{D}}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega}\mathbf{I}_2 \end{bmatrix}. \quad (4.98)$$

By solving the system of equations for a sum rule order four of $\tilde{\mathbf{P}}$, a sum rule order two of \mathbf{P} , and for a vanishing moment order one of both $\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}$, we have 17 free parameters which are $b_{21}, n_{11}, n_{22}, t_{11}, t_{21}, t_{22}, u_{22}, w_{22}, n_{1,11}, n_{1,12}, n_{1,21}, n_{1,22}, d_{1,11}, d_{1,12},$

$d_{1,22}, n_{1,12}, n_{1,22}$.

$$\begin{aligned}
a &= 0, b_{11} = -\frac{1}{2}t_{22}\frac{8d_{1,11}+3}{t_{11}}, b_{12} = \frac{1}{2}\frac{8d_{1,11}t_{22}+3t_{22}+1}{t_{21}}, b_{22} = 0, c = 0, d_{1,21} = 0, \\
d_{11} &= \frac{1}{8}(32d_{1,11}^2t_{22} + 16d_{1,11}n_{11}t_{11} + 16d_{1,11}n_{11}t_{22} - 16d_{1,11}n_{11} + 20d_{1,11}t_{22} + 8n_{11}t_{11} \\
&\quad + 6n_{11}t_{22} + 8d_{1,11} - 6n_{11} + 3t_{22} + 3)/(t_{11}(2d_{1,11} + 1)), \\
d_{12} &= -\frac{1}{4}\frac{8d_{1,11}t_{22}+3t_{22}+1}{t_{21}}, \\
d_{21} &= -\frac{1}{4}-32d_{1,11}n_{22}t_{11}t_{21}u_{22} - 32d_{1,11}n_{22}t_{21}t_{22}u_{22} \\
&\quad + 8b_{21}d_{1,11}t_{11}t_{22} + 32d_{1,11}n_{22}t_{21}u_{22} + 16d_{1,11}t_{11}t_{21}u_{22} + 16d_{1,11}t_{21}t_{22}u_{22} \\
&\quad - 16n_{22}t_{11}t_{21}u_{22} - 12n_{22}t_{21}t_{22}u_{22} + 3b_{21}t_{11}t_{22} - 16d_{1,11}t_{21}u_{22} + 12n_{22}t_{21}u_{22} \\
&\quad + 8t_{11}t_{21}u_{22} + 6t_{21}t_{22}u_{22} + b_{21}t_{11} - 6t_{21}u_{22}/u_{22}(8d_{1,11}t_{22} + 3t_{22} + 1)t_{11}, \\
d_{22} &= \frac{1}{2}, g = 0, h = -\frac{1}{3}, n_{12} = \frac{1}{8}\frac{8d_{1,11}t_{22}+3t_{22}+1)(2n_{11}-1)}{t_{21}(2d_{1,11}+1)}, \\
n_{21} &= 2t_{21}\frac{4d_{1,11}n_{22}-2d_{1,11}+2n_{22}-1}{8d_{1,11}t_{22}+3t_{22}+1}, t_{12} = t_{11}\frac{8d_{1,11}t_{22}+3t_{22}+1}{t_{21}(8d_{1,11}+3)}, u_{11} = \frac{1}{2}, u_{12} = 0, \\
u_{21} &= 0, w_{11} = \frac{1}{8d_{1,11}}, w_{21} = -\frac{1}{2}\frac{(2d_{1,11}+1)t_{21}}{d_{1,11}(8d_{1,11}t_{22}+3t_{22}+1)}, x = 0, z = 0. \\
w_{12} &= -\frac{1}{4}8b_{21}d_{1,11}n_{1,21}t_{11}t_{22}w_{22} + 3b_{21}n_{1,21}t_{11}t_{22}w_{22} \\
&\quad + 8d_{1,11}n_{1,21}t_{11}t_{21}u_{22} + 8d_{1,11}n_{1,21}t_{21}t_{22}u_{22} + 2b_{21}d_{1,11}t_{11}t_{21} + b_{21}n_{1,21}t_{11}w_{22} \\
&\quad - 8d_{1,11}n_{1,21}t_{21}u_{22} + 4n_{1,21}t_{11}t_{21}u_{22} + 3n_{1,21}t_{21}t_{22}u_{22} + b_{21}t_{11}t_{21} - 3n_{1,21}t_{21} \\
&\quad u_{22}/(2d_{1,11}+1)t_{21}b_{21}n_{1,21}t_{11},
\end{aligned}$$

If we choose

$$\begin{aligned}
&[b_{21}, d_{1,11}, n_{11}, n_{1,11}, n_{1,21}, n_{22}, t_{11}, t_{21}, t_{22}, u_{22}, w_{22}, n_{1,12}, n_{1,22}, d_{1,12}, d_{1,22}, n_{1,12}, n_{1,22}] = \left[\frac{-167}{317}, \right. \\
&\left. \frac{493}{419}, \frac{2}{85}, \frac{1257}{281}, \frac{-83}{75}, -\frac{333}{403}, -\frac{570}{379}, -\frac{502}{409}, \frac{115}{496}, \frac{8}{181}, \frac{137}{644}, \frac{168}{463}, \frac{-20}{473}, \frac{103}{284}, \frac{151}{689}, \frac{13}{64}, \frac{16}{43}\right].
\end{aligned}$$

By using the smoothness formula, we can obtain $\tilde{\Phi} \in W^{3.5}$ and $\Phi \in W^{0.5217}$.

The lowpass analysis mask \mathbf{P} and the lowpass synthesis mask $\tilde{\mathbf{P}}$ will be as follows:

$$\mathbf{P}(\omega) = \begin{bmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned}
P_{11}(\omega) &= 0.125(e^{-2i\omega} + e^{2i\omega}) + 0.01834725537(e^{-i\omega} + e^{i\omega}) + 0.25 - 0.2316527446(e^{-3i\omega} + e^{3i\omega}), \\
P_{12}(\omega) &= -0.05380029202 + 0.01345007300(e^{-i\omega} + e^{i\omega}) + 0.02690014601(e^{-2i\omega} + e^{2i\omega}) \\
&\quad - 0.01345007300(e^{-3i\omega} + e^{3i\omega}), \\
P_{21}(\omega) &= -0.03279205050(e^{-2i\omega} + e^{2i\omega}) + 0.3186100719(e^{-i\omega} + e^{i\omega}) - 1.832619492 \\
&\quad + 0.06976549456(e^{-3i\omega} + e^{3i\omega}), \\
P_{22}(\omega) &= -0.05564866798(e^{-2i\omega} + e^{2i\omega}) - 0.0005278712354(e^{-i\omega} + e^{i\omega}) - 1.260959071 \\
&\quad - 0.03652774488(e^{-3i\omega} + e^{3i\omega}).
\end{aligned}$$

and

$$\tilde{\mathbf{P}}(\omega) = \begin{bmatrix} \tilde{P}_{11}(\omega) & \tilde{P}_{12}(\omega) \\ \tilde{P}_{21}(\omega) & \tilde{P}_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned}
\tilde{P}_{11}(\omega) &= 0.0625(e^{-2i\omega} + e^{2i\omega}) + 0.375 + 0.25(e^{-i\omega} + e^{i\omega}), \\
\tilde{P}_{12}(\omega) &= 0, \\
\tilde{P}_{21}(\omega) &= 0.09903822944(e^{-2i\omega} + e^{2i\omega}) - 0.1980764589, \\
\tilde{P}_{22}(\omega) &= -0.03625875174(e^{-2i\omega} + e^{2i\omega}) - 0.02831860845 + 0.02209944751(e^{-i\omega} + e^{i\omega}).
\end{aligned}$$

Thus, the resulting Φ is supported on $[-3, 3]$. The highpass analysis mask \mathbf{Q} and the highpass synthesis mask $\tilde{\mathbf{Q}}$ will be as follows:

$$\mathbf{Q}_1^{(1)}(\omega) = \begin{bmatrix} Q_{11}^{(1)}(\omega) & Q_{12}^{(1)}(\omega) \\ Q_{21}^{(1)}(\omega) & Q_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}
Q_{11}^{(1)}(\omega) &= 0.1362766092(e^{-3i\omega} + e^{3i\omega}) - 0.05349313887(e^{-2i\omega} + e^{2i\omega}) \\
&\quad - 5.292903315(e^{-i\omega} + e^{i\omega}) + 0.8930137223, \\
Q_{12}^{(1)}(\omega) &= -1.533165616(e^{-3i\omega} + e^{3i\omega}) - 0.3021997879(e^{-2i\omega} + e^{2i\omega}) \\
&\quad - 2.137565191(e^{-i\omega} + e^{i\omega}) - 0.6043995757, \\
Q_{21}^{(1)}(\omega) &= 0.08007592886(e^{-3i\omega} + e^{3i\omega}) - 0.06836498218(e^{-2i\omega} + e^{2i\omega}) \\
&\quad - 0.01021028291(e^{i\omega} - e^{-i\omega}) + 0.1367299644, \\
Q_{22}^{(1)}(\omega) &= 0.3401890887(e^{-3i\omega} + e^{3i\omega}) - 0.01329275962(e^{-2i\omega} + e^{2i\omega}) \\
&\quad + 1.627711795(e^{-i\omega} + e^{i\omega}) + 0.9734144808.
\end{aligned}$$

$$\mathbf{Q}_2^{(2)}(\omega) = \begin{bmatrix} Q_{11}^{(2)}(\omega) & Q_{12}^{(2)}(\omega) \\ Q_{21}^{(2)}(\omega) & Q_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned} Q_{11}^{(2)}(\omega) &= 0.6062373225e^{-i\omega} - 0.1968813387(e^{i\omega} + e^{-3i\omega}) - 0.1062373225(1 + e^{-2i\omega}), \\ Q_{12}^{(2)}(\omega) &= -0.06724825955(e^{i\omega} + e^{-3i\omega}) - 1.344965191e^{-i\omega} + 0.1344965191(1 + e^{-2i\omega}), \\ Q_{21}^{(2)}(\omega) &= -0.03956859174(e^{i\omega} + e^{-3i\omega}) - 0.07913718348e^{-i\omega} - 0.1361356726(1 + e^{-2i\omega}), \\ Q_{22}^{(2)}(\omega) &= -1.440967095(e^{i\omega} + e^{-3i\omega}) - 18.81934190e^{-i\omega} - 1.619191409(1 + e^{-2i\omega}). \end{aligned}$$

and

$$\tilde{\mathbf{Q}}_1^{(1)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(1)}(\omega) & \tilde{Q}_{12}^{(1)}(\omega) \\ \tilde{Q}_{21}^{(1)}(\omega) & \tilde{Q}_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned} \tilde{Q}_{11}^{(1)}(\omega) &= 0.7606591630 + 0.5063190274(e^{2i\omega} + e^{-2i\omega}) + 0.5317258883(e^{i\omega} + e^{-i\omega}), \\ \tilde{Q}_{12}^{(1)}(\omega) &= 0.1805059716(e^{2i\omega} + e^{-2i\omega}) + 0.1696044732 + 0.1100170870(e^{i\omega} + e^{-i\omega}), \\ \tilde{Q}_{21}^{(1)}(\omega) &= 0.4203045639(e^{2i\omega} + e^{-2i\omega}) + 0.2269171962 - 0.2254983405(e^{i\omega} + e^{-i\omega}), \\ \tilde{Q}_{22}^{(1)}(\omega) &= 0.7331052930 - 0.1745163562(e^{2i\omega} + e^{-2i\omega}) + 0.1063664596(e^{i\omega} + e^{-i\omega}). \end{aligned}$$

$$\tilde{\mathbf{Q}}_2^{(2)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(2)}(\omega) & \tilde{Q}_{12}^{(2)}(\omega) \\ \tilde{Q}_{21}^{(2)}(\omega) & \tilde{Q}_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned} \tilde{Q}_{11}^{(2)}(\omega) &= 2.388309115 + 2.043806928e^{-i\omega} + 3.994659101e^{-2i\omega} \\ &\quad - 1.449966807(e^{2i\omega} + e^{-4i\omega}) + 0.5219034638(e^{i\omega} + e^{-3i\omega}), \\ \tilde{Q}_{12}^{(2)}(\omega) &= 0.6404158360 + 0.5888639357e^{-i\omega} + 1.304523005e^{-2i\omega} \\ &\quad + 0.4830770371(e^{2i\omega} + e^{-4i\omega}) - 0.2944319679(e^{i\omega} + e^{-3i\omega}), \\ \tilde{Q}_{21}^{(2)}(\omega) &= 0.5253841718 - 0.4328178490e^{-i\omega} + 1.969298059e^{-2i\omega} - \\ &\quad 0.9448017230(e^{2i\omega} + e^{-4i\omega}) - 0.2164089245(e^{i\omega} + e^{-3i\omega}), \end{aligned}$$

$$\begin{aligned}\tilde{Q}_{22}^{(2)}(\omega) = & 0.2754640218 + 0.6024976056e^{-i\omega} + 1.262951382e^{-2i\omega} \\ & - 0.3260927819(e^{2i\omega} + e^{-4i\omega}) - 0.1987511972 + e^{-3i\omega}.\end{aligned}$$

Remark: when the matrices \mathbf{D}_1 and \mathbf{N}_1 are zero matrices, So the 3-step algorithm type II is reduced to a 2-step algorithm type II. Thus, the 2-step decomposition algorithm type II is give by (4.85) with $\tilde{\mathbf{v}} = \mathbf{v}''$ and $\tilde{\mathbf{f}} = \mathbf{f}''$ and the 2-step reconstruction algorithm type II is give by (4.86) with $\mathbf{v}'' = \tilde{\mathbf{v}}$ and $\mathbf{f}'' = \tilde{\mathbf{f}}$. For 2-step algorithm of this type, the multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ are given by (4.95) and (4.96) with $\mathbf{D}_2(\omega) = \tilde{\mathbf{D}}_2(\omega) = \mathbf{I}_6$ and $\mathbf{D}_0(\omega), \tilde{\mathbf{D}}_0(\omega)$ and $\mathbf{D}_1(\omega), \tilde{\mathbf{D}}_1(\omega)$ are the same as in the 3-step algorithm of this type.

Table 4.1: The Smoothness of biorthogonal multiwavelets/multiple bi-frames in 1-D.

Multiresolution Algorithms	Sobolev smoothness estimate for Φ	Sobolev smoothness estimate for $\tilde{\Phi}$
2-Step bi-multiwavelets	0.0073	0.2098
3-Step bi-multiwavelets	0.3283	1.7107
2-Step type I multiple bi-frames	-	0.3547
3-Step type I multiple bi-frames	1.2673	2
2-Step type II multiple bi-frames	1.5	2.6540
3-Step type II multiple bi-frames	0.5217	3.5

Table 4.2: The Smoothness of biorthogonal wavelets/frames in 1-D.

Multiresolution Algorithms	Sobolev smoothness estimate for ϕ	Sobolev smoothness estimate for $\tilde{\phi}$
2-Step bi-wavelets	1.5	0.4408
3-Step bi-wavelets	0.12976	3.5
2-Step type I bi-frames	1.5	0.4408
2-Step type II bi-frames	1.5	2
3-Step type II bi-frames	1.86992	3.5

The wavelet delicate thresholding system presented by Coifman and Donoho [22]. They have introduced a general edge T

$$T = \sqrt{2 \cdot \sigma^2 \cdot \log N}$$

where σ is the variance and N is the total number of pixels. There are three steps required to remove noise from the highpass coefficients. First, determine the wavelet coefficients.

Then, by threshold process we can modify the details coefficients. Finally, restore the denoised data. Let x be the wavelet coefficient, T is the threshold, and \hat{x} be the changed coefficient. Hard-thresholding method removes the coefficients which are having values less than T and keeps the other coefficients. Hard-thresholding rule defined by

$$\hat{x}_T^H = \begin{cases} x, & |x| \geq T \\ 0, & |x| < T \end{cases}$$

and soft-thresholding rule defined by

$$\hat{x}_T^S = \begin{cases} \text{sgn}(x)(|x| - T), & |x| \geq T \\ 0, & |x| < T \end{cases}$$

We will introduce the denoising algorithm. Let $\{\mathbf{u}_k^0\}_k \in \mathbf{Z}^2$ be the initial input. The multiple bi-frame denoising algorithm is

$$\mathbf{L}_n = \sum_{\mathbf{k} \in \mathbf{Z}^2} \mathbf{u}_{\mathbf{k}+\mathbf{n}}^0 \mathbf{P}_k^T, \quad \mathbf{H}_n^{(\ell)} = \sum_{\mathbf{k} \in \mathbf{Z}^2} \sum_{\ell=1}^L \mathbf{u}_{\mathbf{k}+\mathbf{n}}^0 \mathbf{Q}_k^{(\ell)T}, \quad \mathbf{u}_k^1 = \sum_{\mathbf{n} \in \mathbf{Z}^2} \mathbf{L}_n \tilde{\mathbf{P}}_{\mathbf{k}-\mathbf{n}} + \sum_{\ell=1}^L \sum_{\mathbf{n} \in \mathbf{Z}^2} (\mathbf{H}_n^{(\ell)}) \tilde{\mathbf{Q}}_{\mathbf{k}-\mathbf{n}}^{(\ell)}. \quad (4.99)$$

\mathbf{L} is the lowpass output and \mathbf{H} is the highpass output. \mathbf{u}_k is called the denoising data of the original data with noise. The multiple frame algorithm can be used for curve noise-removing. We show in Fig. 4.9 how to remove noise from a curve by applying multiple frame algorithm 15 times and hard and soft thresholding process for noise-removing after adding Gaussian noise to the original curve. In Fig.4.9, the left column shows the original curves, the middle column shows the noised curves, and the right column shows the denoised curves.

4.2.7 Uniform Symmetry

In this subsection, we follow the work in (the scalar case) in the paper [3]. We derive the proposition in (the vector case) which states that the framelets which are obtained by type II multiple bi-frame multiresolution algorithms have uniform symmetry.

In general case, the type II frame multifilter banks corresponding to algorithms with more steps can be written in general forms as the following. Let $\mathbf{E}(\omega)$ be the matrix of the form:

$$\mathbf{E}(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{L}_3(e^{-i\omega}) \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{L}_4(e^{-i\omega}) \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \quad (4.100)$$

and $\tilde{\mathbf{E}}(\omega) = (\mathbf{E}(\omega)^{-1})^*$ is the matrix of the form:

$$\tilde{\mathbf{E}}(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ -\mathbf{L}_3(e^{i\omega}) & -\mathbf{L}_4(e^{i\omega}) & \mathbf{I}_2 \end{bmatrix}, \quad (4.101)$$

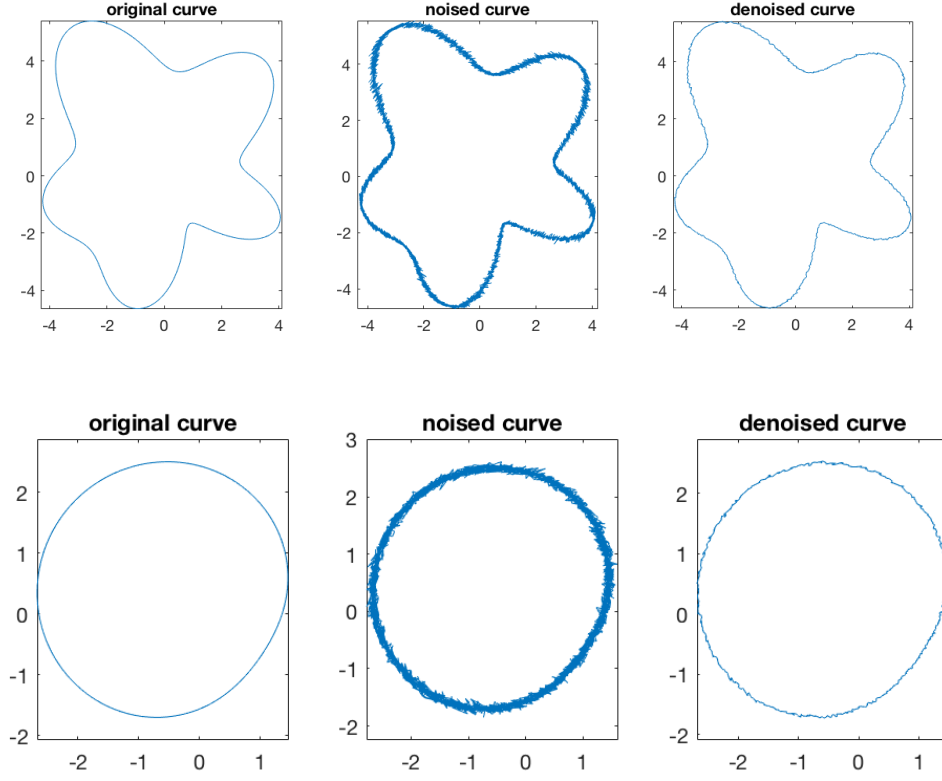


Figure 4.9: Left column: original curves; middle column: noised curves; right column: denoised curves.

where $\mathbf{L}_3(e^{-i\omega})$ and $\mathbf{L}_4(e^{-i\omega})$ are Laurent matrix polynomials satisfying

$$\mathbf{L}_3(e^{i\omega}) = e^{-i\omega} \mathbf{L}_3(e^{-i\omega}), \quad \mathbf{L}_4(e^{i\omega}) = e^{-i\omega} \mathbf{L}_4(e^{-i\omega}), \quad (4.102)$$

then $\mathbf{L}_3(e^{-i\omega})$ and $\mathbf{L}_4(e^{-i\omega})$ are Laurent matrix polynomials of the form:

$$\mathbf{L}(e^{-i\omega}) = (1 + e^{i\omega})\mathbf{S}_1 + (e^{-i\omega} + e^{2i\omega})\mathbf{S}_2 + \dots + (e^{-(m-1)i\omega} + e^{mi\omega})\mathbf{S}_m, \quad (4.103)$$

where m is a positive integer and \mathbf{S}_s are real-valued 2×2 matrices, $s = 1, 2, \dots, m$. Then the type II frame multifilter banks corresponding to algorithms with K ($K \geq 1$) steps can be written as the following

$$\begin{bmatrix} \mathbf{P}(\omega) \\ \mathbf{Q}^{(1)}(\omega) \\ \mathbf{Q}^{(2)}(\omega) \end{bmatrix} = \mathbf{D}_{K-1}(2\omega) \mathbf{D}_{K-2}(2\omega) \dots \mathbf{D}_1(2\omega) \mathbf{D}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega} \mathbf{I}_2 \end{bmatrix}, \quad (4.104)$$

$$\begin{bmatrix} \tilde{\mathbf{P}}(\omega) \\ \tilde{\mathbf{Q}}^{(1)}(\omega) \\ \tilde{\mathbf{Q}}^{(2)}(\omega) \end{bmatrix} = \frac{1}{2} \tilde{\mathbf{D}}_{K-1}(2\omega) \tilde{\mathbf{D}}_{K-2}(2\omega) \dots \tilde{\mathbf{D}}_1(2\omega) \tilde{\mathbf{D}}_0(2\omega) \begin{bmatrix} \mathbf{I}_2 \\ e^{-i\omega} \mathbf{I}_2 \end{bmatrix}, \quad (4.105)$$

where the matrices $\mathbf{D}_0(\omega)$ and $\tilde{\mathbf{D}}_0(\omega)$ are given in (4.95) and (4.96) respectively, each $\mathbf{D}_k(\omega)$, $1 \leq k \leq K-1$, is a matrix $\mathbf{E}(\omega)$ or $\tilde{\mathbf{E}}(\omega)$, where $\tilde{\mathbf{D}}_k(\omega) = (\mathbf{D}_k(\omega)^{-1})^*$.

Proposition 2. Let $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}\}$ be the biorthogonal frame multifilter banks defined by (4.104) and (4.105). Then

$$\begin{aligned} \mathbf{P}(-\omega) &= \mathbf{P}(\omega), \quad \mathbf{Q}^{(1)}(-\omega) = \mathbf{Q}^{(1)}(\omega), \quad \mathbf{Q}^{(2)}(-\omega) = e^{i2\omega} \mathbf{Q}^{(2)}(\omega), \quad \tilde{\mathbf{P}}(-\omega) = \tilde{\mathbf{P}}(\omega), \\ \tilde{\mathbf{Q}}^{(1)}(-\omega) &= \tilde{\mathbf{Q}}^{(1)}(\omega), \quad \tilde{\mathbf{Q}}^{(2)}(-\omega) = e^{i2\omega} \tilde{\mathbf{Q}}^{(2)}(\omega). \end{aligned} \quad (4.106)$$

Therefore, the associated multiscaling function Φ , $\tilde{\Phi}$ and multiwavelets $\Psi^{(\ell)}$, $\tilde{\Psi}^{(\ell)}$, $\ell = 1, 2$, satisfy

$$\begin{aligned} \Phi(x) &= \Phi(-x), \quad \Psi^{(1)}(x) = \Psi^{(1)}(-x), \quad \Psi^{(2)}(x) = \Psi^{(2)}(1-x), \quad \tilde{\Phi}(x) = \tilde{\Phi}(-x), \\ \tilde{\Psi}^{(1)}(x) &= \tilde{\Psi}^{(1)}(-x), \quad \tilde{\Psi}^{(2)}(x) = \tilde{\Psi}^{(2)}(1-x). \end{aligned} \quad (4.107)$$

Proof. From (4.100), we can easily verify that

$$\begin{aligned} \mathbf{D}_K(-\omega) &= \text{diag}(\mathbf{I}_2, \mathbf{I}_2, e^{i\omega} \mathbf{I}_2) \mathbf{D}_K(\omega) \text{diag}(\mathbf{I}_2, \mathbf{I}_2, e^{-i\omega} \mathbf{I}_2), \\ \mathbf{D}_0(-\omega) &= \text{diag}(\mathbf{I}_2, \mathbf{I}_2, e^{i\omega} \mathbf{I}_2) \mathbf{D}_0(\omega) \text{diag}(\mathbf{I}_2, e^{-i\omega} \mathbf{I}_2), \end{aligned} \quad (4.108)$$

where $1 \leq k \leq K-1$. (4.108) implies

$$\begin{bmatrix} \mathbf{P}(-\omega) \\ \mathbf{Q}^{(1)}(-\omega) \\ \mathbf{Q}^{(2)}(-\omega) \end{bmatrix} = \text{diag}(\mathbf{I}_2, e^{i2\omega} \mathbf{I}_2, e^{i2\omega} \mathbf{I}_2) \begin{bmatrix} \mathbf{P}(\omega) \\ \mathbf{Q}^{(1)}(\omega) \\ \mathbf{Q}^{(2)}(\omega) \end{bmatrix}.$$

Since $\tilde{\mathbf{D}}_K(\omega)$ and $\tilde{\mathbf{D}}_0(\omega)$ satisfy (4.108), then we can obtain the symmetry of $\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}$.

From $\hat{\Phi}(\omega) = \prod_{j=1}^{\infty} \mathbf{P}(2^{-j}\omega)\hat{\Phi}(0)$ and $\mathbf{P}(-\omega) = \mathbf{P}(\omega)$, we have

$$\hat{\Phi}(-\omega) = \prod_{j=1}^{\infty} \mathbf{P}(-2^{-j}\omega)\hat{\Phi}(0) = \prod_{j=1}^{\infty} \mathbf{P}(2^{-j}\omega)\hat{\Phi}(0) = \hat{\Phi}(\omega).$$

Thus $\Phi(-x) = \Phi(x)$.

From $\hat{\Psi}^{(\ell)}(\omega) = \mathbf{Q}^{(\ell)}(\frac{\omega}{2})\hat{\Phi}(\frac{\omega}{2})$ and $\mathbf{Q}^{(\ell)}(-\omega) = e^{i2\omega}\mathbf{Q}^{(\ell)}(\omega)$, we have

$$\hat{\Psi}^{(\ell)}(-\omega) = \mathbf{Q}^{(\ell)}(-\frac{\omega}{2})\hat{\Phi}(-\frac{\omega}{2}) = e^{i\omega}\mathbf{Q}^{(\ell)}(\frac{\omega}{2})\hat{\Phi}(\frac{\omega}{2}) = e^{i\omega}\hat{\Psi}^{(\ell)}(\omega).$$

Thus $\Psi^{(\ell)}(-x) = \Psi^{(\ell)}(x+1)$.

Similarly, we can proof the symmetry of $\tilde{\Phi}$ and $\tilde{\Psi}^{(\ell)}$.

From (4.101), we can easily verify that

$$\begin{aligned}\tilde{\mathbf{D}}_K(-\omega) &= \text{diag}(\mathbf{I}_2, \mathbf{I}_2, e^{i\omega}\mathbf{I}_2)\tilde{\mathbf{D}}_K(\omega)\text{diag}(\mathbf{I}_2, \mathbf{I}_2, e^{-i\omega}\mathbf{I}_2), \\ \tilde{\mathbf{D}}_0(-\omega) &= \text{diag}(\mathbf{I}_2, \mathbf{I}_2, e^{i\omega}\mathbf{I}_2)\tilde{\mathbf{D}}_0(\omega)\text{diag}(\mathbf{I}_2, \mathbf{I}_2, e^{-i\omega}\mathbf{I}_2),\end{aligned}\tag{4.109}$$

where $1 \leq k \leq K-1$. (4.109) implies

$$\begin{bmatrix} \tilde{\mathbf{P}}(-\omega) \\ \tilde{\mathbf{Q}}^{(1)}(-\omega) \\ \tilde{\mathbf{Q}}^{(2)}(-\omega) \end{bmatrix} = \text{diag}(\mathbf{I}_2, \mathbf{I}_2, e^{i2\omega}\mathbf{I}_2) \begin{bmatrix} \tilde{\mathbf{P}}(\omega) \\ \tilde{\mathbf{Q}}^{(1)}(\omega) \\ \tilde{\mathbf{Q}}^{(2)}(\omega) \end{bmatrix}.$$

Since $\mathbf{D}_K(\omega)$ and $\mathbf{D}_0(\omega)$ satisfy (4.108), then we can obtain the symmetry of $\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}$.

From $\hat{\Phi}(\omega) = \prod_{j=1}^{\infty} \tilde{\mathbf{P}}(2^{-j}\omega)\hat{\Phi}(0)$ and $\tilde{\mathbf{P}}(-\omega) = \tilde{\mathbf{P}}(\omega)$, we have

$$\hat{\Phi}(-\omega) = \prod_{j=1}^{\infty} \tilde{\mathbf{P}}(-2^{-j}\omega)\hat{\Phi}(0) = \prod_{j=1}^{\infty} \tilde{\mathbf{P}}(2^{-j}\omega)\hat{\Phi}(0) = \hat{\Phi}(\omega).$$

Thus $\tilde{\Phi}(-x) = \tilde{\Phi}(x)$.

From $\hat{\tilde{\Psi}}^{(\ell)}(\omega) = \tilde{\mathbf{Q}}^{(\ell)}(\frac{\omega}{2})\hat{\Phi}(\frac{\omega}{2})$ and $\tilde{\mathbf{Q}}^{(\ell)}(-\omega) = e^{i2\omega}\tilde{\mathbf{Q}}^{(\ell)}(\omega)$, we have

$$\hat{\tilde{\Psi}}^{(\ell)}(-\omega) = \tilde{\mathbf{Q}}^{(\ell)}(-\frac{\omega}{2})\hat{\Phi}(-\frac{\omega}{2}) = e^{i\omega}\tilde{\mathbf{Q}}^{(\ell)}(\frac{\omega}{2})\hat{\Phi}(\frac{\omega}{2}) = e^{i\omega}\hat{\tilde{\Psi}}^{(\ell)}(\omega).$$

Thus $\tilde{\Psi}^{(\ell)}(-x) = \tilde{\Psi}^{(\ell)}(x+1)$.

Chapter 5

Two-dimensional Biorthogonal Multiwavelets and Multiresolution Algorithms

5.1 Introduction

In this chapter, the work will be an extension of the work in [66] from (the scalar case) to (the vector case), which studies the biorthogonal wavelets for triangular mesh-based surface multiresolution (multiscale) processing. The multifilter bank combines of a lowpass filter and three highpass filters that means $\ell = 3$.

Recently hexagonal data/image processing has important advantages. If we compare hexagonal lattice with square lattice, we will see that hexagonal lattice has more advantages. Hexagonal lattice has 6-fold symmetry (6 axes of symmetry) while square lattice has 4-fold symmetry. Thus, hexagonal lattice has high symmetry which will make the multifilter banks along it also have 6-fold symmetry. See the pictures in Fig 5.1 and 5.3 for a square, hexagonal lattice and 6-fold symmetry. The high symmetry make simpler algorithms for the construction of compactly supported biorthogonal wavelet. Also, it plays an important role for the applications of texture segmentation of hexagonal data.

Let $\mathbf{k} = (k_1, k_2) \in \mathbf{Z}^2$, and $\boldsymbol{\omega} = (\omega_1, \omega_2)$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. The multi-indices $\mathbf{k} = (k_1, k_2)$ are the indices for the coefficients $\mathbf{P}_{\mathbf{k}}$, $\mathbf{Q}_{\mathbf{k}}^{(\ell)}$, $1 \leq \ell \leq 3$.

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbf{Z}^2} \mathbf{P}_{\mathbf{k}} \Phi(2\mathbf{x} - \mathbf{k}), \quad \mathbf{P}(\boldsymbol{\omega}) = \frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^2} e^{-i\mathbf{k}\boldsymbol{\omega}} \mathbf{P}_{\mathbf{k}},$$

$$\tilde{\Phi}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbf{Z}^2} \tilde{\mathbf{P}}_{\mathbf{k}} \tilde{\Phi}(2\mathbf{x} - \mathbf{k}), \quad \tilde{\mathbf{P}}(\boldsymbol{\omega}) = \frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^2} e^{-i\mathbf{k}\boldsymbol{\omega}} \tilde{\mathbf{P}}_{\mathbf{k}},$$

$$\Psi^{(\ell)}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbf{Z}^2} \mathbf{Q}_{\mathbf{k}}^{(\ell)} \Phi(2\mathbf{x} - \mathbf{k}), \quad \mathbf{Q}^{(\ell)}(\boldsymbol{\omega}) = \frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^2} e^{-i\mathbf{k}\boldsymbol{\omega}} \mathbf{Q}_{\mathbf{k}}^{(\ell)}.$$

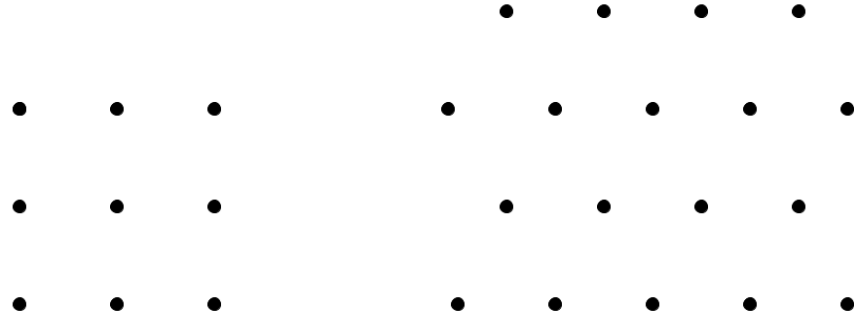


Figure 5.1: Left: Square lattice; Right: Hexagonal lattice.

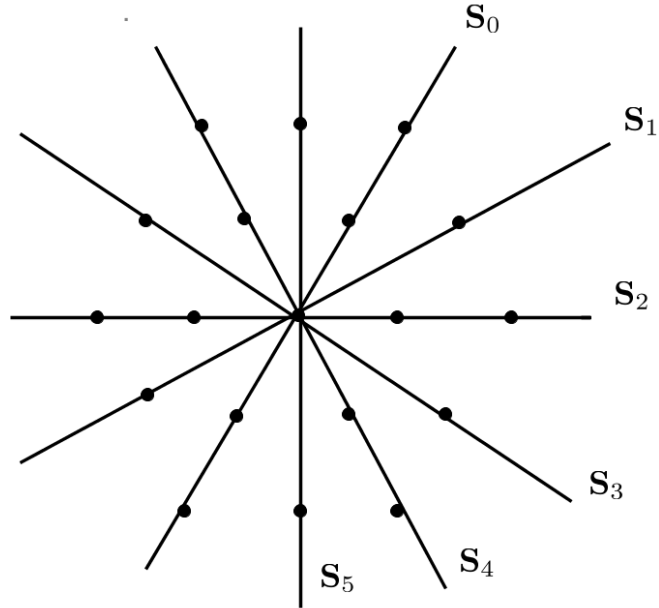


Figure 5.2: 6 symmetric lines (axes).

$$\tilde{\Psi}^{(\ell)}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbf{Z}^2} \tilde{\mathbf{Q}}_{\mathbf{k}}^{(\ell)} \tilde{\Phi}(2\mathbf{x} - \mathbf{k}), \quad \tilde{\mathbf{Q}}^{(\ell)}(\boldsymbol{\omega}) = \frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^2} e^{-i\mathbf{k}\boldsymbol{\omega}} \tilde{\mathbf{Q}}_{\mathbf{k}}^{(\ell)}.$$

The functions are 2×1 vectors and the masks are 2×2 matrices.

FIR multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}\}$ are said to be biorthog-

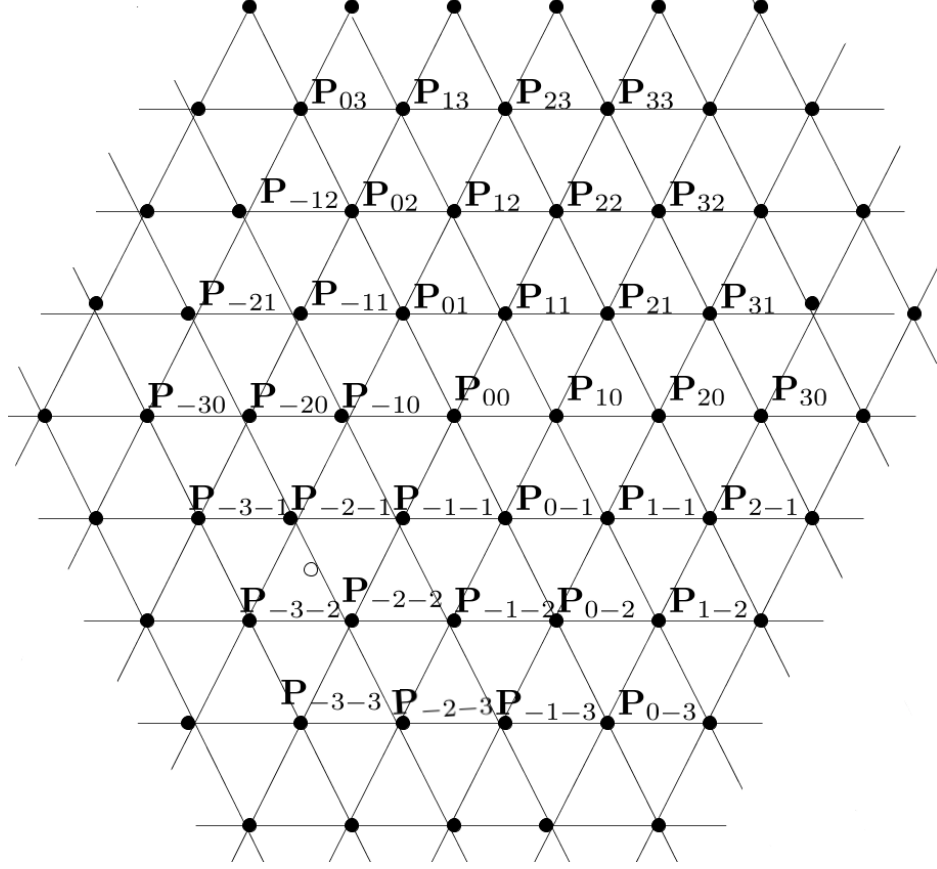


Figure 5.3: Indices for the lowpass multifilter bank coefficients \mathbf{P}_{k_1, k_2}

onal if they satisfy the biorthogonality conditions:

$$\left\{ \begin{array}{l} \sum_{j=0}^3 \mathbf{P}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j) \tilde{\mathbf{P}}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j)^* = \mathbf{I}_2, \\ \sum_{j=0}^3 \mathbf{P}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j) \tilde{\mathbf{Q}}^{(\ell)}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j)^* = \mathbf{0}_2, \\ \sum_{j=0}^3 \mathbf{Q}^{(\ell)}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j) \tilde{\mathbf{P}}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j)^* = \mathbf{0}_2, \\ \sum_{j=0}^3 \mathbf{Q}^{(\ell)}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j) \tilde{\mathbf{Q}}^{(\ell')}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j)^* = \delta_{\ell, \ell'} = \begin{cases} \mathbf{I}_2, & \ell = \ell' \\ \mathbf{0}_2, & \ell \neq \ell' \end{cases} \end{array} \right. \quad (5.1)$$

which is equivalent to the perfect reconstruction conditions

$$\tilde{\mathbf{P}}(\boldsymbol{\omega})^* \mathbf{P}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j) + \sum_{\ell=1}^3 \tilde{\mathbf{Q}}^{(\ell)}(\boldsymbol{\omega})^* \mathbf{Q}^{(\ell)}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j) = \begin{cases} \mathbf{I}_2, & j = 0, \\ \mathbf{0}_2, & 1 \leq j \leq 3, \end{cases} \quad (5.2)$$

where $1 \leq \ell, \ell' \leq 3$, $\boldsymbol{\eta}_0 = (0, 0)$, $\boldsymbol{\eta}_1 = (-1, -1)$, $\boldsymbol{\eta}_2 = (1, 0)$, $\boldsymbol{\eta}_3 = (0, 1)$.

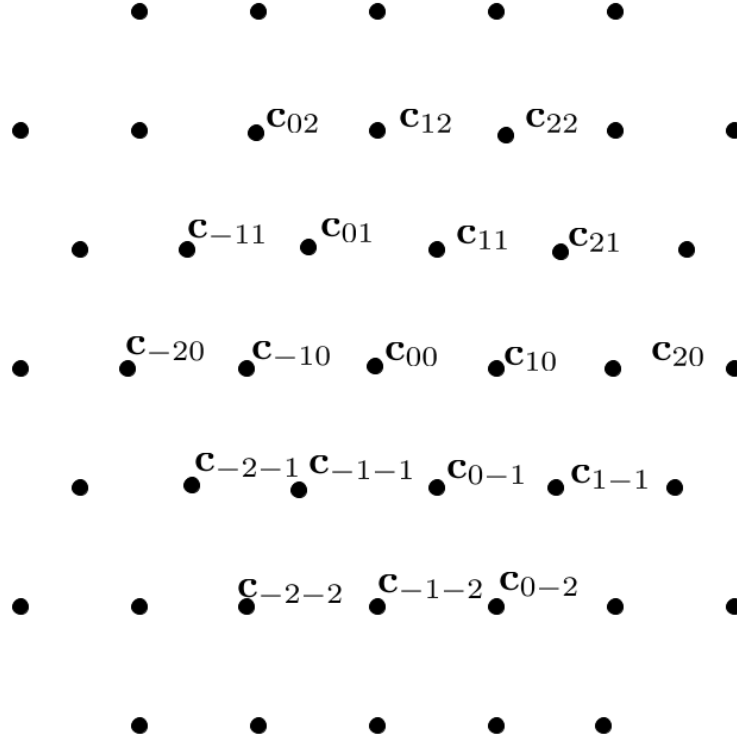


Figure 5.4: Indices for the data $\mathbf{c}_{(k_1, k_2)}$

Let $\mathbf{I}_0(\boldsymbol{\omega})$ be 1-tap initial multifilter bank, which is 8×2 matrix and has 6 fold symmetry, is defined by

$$\mathbf{I}_0(\boldsymbol{\omega}) = [\mathbf{I}_2, e^{i(\omega_1 + \omega_2)} \mathbf{I}_2, e^{-i\omega_1} \mathbf{I}_2, e^{-i\omega_2} \mathbf{I}_2]^T. \quad (5.3)$$

For a multifilter set $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}\}$, with $\mathbf{Q}^{(0)}(\boldsymbol{\omega}) = \mathbf{P}(\boldsymbol{\omega})$, write $\mathbf{Q}^{(\ell)}, 1 \leq \ell \leq 3$ as

$$\mathbf{Q}^{(\ell)} = \frac{1}{2}(\mathbf{Q}_0^{(\ell)}(2\boldsymbol{\omega}) + \mathbf{Q}_1^{(\ell)}(2\boldsymbol{\omega})e^{i(\omega_1 + \omega_2)} + \mathbf{Q}_2^{(\ell)}(2\boldsymbol{\omega})e^{-i\omega_1} + \mathbf{Q}_3^{(\ell)}(2\boldsymbol{\omega})e^{-i\omega_2}), \quad (5.4)$$

where $\mathbf{Q}_k^{(\ell)}(\boldsymbol{\omega}), 0 \leq k \leq 3$ are matrix trigonometric polynomials and $\boldsymbol{\omega} = (\omega_1, \omega_2)$.

The **polyphase matrix** of the multifilter set $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}\}$ is the 8×8 matrix $\mathbf{V}(\boldsymbol{\omega})$ given by

$$\mathbf{V}(\boldsymbol{\omega}) = [\mathbf{Q}_k^{(\ell)}(\boldsymbol{\omega})]_{0 \leq \ell \leq 3, 0 \leq k \leq 3}, \quad (5.5)$$

and

$$\tilde{\mathbf{V}}(\boldsymbol{\omega}) = [\tilde{\mathbf{Q}}_k^{(\ell)}(\boldsymbol{\omega})]_{0 \leq \ell \leq 3, 0 \leq k \leq 3}. \quad (5.6)$$

$$\mathbf{V}(\omega) := \begin{bmatrix} \mathbf{P}_0(\omega) & \mathbf{P}_1(\omega) & \mathbf{P}_2(\omega) & \mathbf{P}_3(\omega) \\ \mathbf{Q}_0^{(1)}(\omega) & \mathbf{Q}_1^{(1)}(\omega) & \mathbf{Q}_2^{(1)}(\omega) & \mathbf{Q}_3^{(1)}(\omega) \\ \mathbf{Q}_0^{(2)}(\omega) & \mathbf{Q}_1^{(2)}(\omega) & \mathbf{Q}_2^{(2)}(\omega) & \mathbf{Q}_3^{(2)}(\omega) \\ \mathbf{Q}_0^{(3)}(\omega) & \mathbf{Q}_1^{(3)}(\omega) & \mathbf{Q}_2^{(3)}(\omega) & \mathbf{Q}_3^{(3)}(\omega) \end{bmatrix}_{8 \times 8},$$

$$\tilde{\mathbf{V}}(\omega) := \begin{bmatrix} \tilde{\mathbf{P}}_0(\omega) & \tilde{\mathbf{P}}_1(\omega) & \tilde{\mathbf{P}}_2(\omega) & \tilde{\mathbf{P}}_3(\omega) \\ \tilde{\mathbf{Q}}_0^{(1)}(\omega) & \tilde{\mathbf{Q}}_1^{(1)}(\omega) & \tilde{\mathbf{Q}}_2^{(1)}(\omega) & \tilde{\mathbf{Q}}_3^{(1)}(\omega) \\ \tilde{\mathbf{Q}}_0^{(2)}(\omega) & \tilde{\mathbf{Q}}_1^{(2)}(\omega) & \tilde{\mathbf{Q}}_2^{(2)}(\omega) & \tilde{\mathbf{Q}}_3^{(2)}(\omega) \\ \tilde{\mathbf{Q}}_0^{(3)}(\omega) & \tilde{\mathbf{Q}}_1^{(3)}(\omega) & \tilde{\mathbf{Q}}_2^{(3)}(\omega) & \tilde{\mathbf{Q}}_3^{(3)}(\omega) \end{bmatrix}_{8 \times 8}.$$

Then,

$$[\mathbf{P}(\omega), \mathbf{Q}^{(1)}(\omega), \mathbf{Q}^{(2)}(\omega), \mathbf{Q}^{(3)}(\omega)]^T = \frac{1}{2} \mathbf{V}(2\omega) \mathbf{I}_0(\omega), \quad (5.7)$$

The two sets of multifilters $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}\}$ are biorthogonal if and only if

$$\mathbf{V}(\omega) \tilde{\mathbf{V}}(\omega)^* = \mathbf{I}_8, \quad \omega \in \mathbb{R}^2, \quad (5.8)$$

where $\mathbf{V}(\omega)^*$ denotes the complex conjugate and transpose of the matrix $\mathbf{V}(\omega)$ and the polyphase matrices $\mathbf{V}(\omega)$, $\tilde{\mathbf{V}}(\omega)$ defined as above.

We know that algorithms can be given by templates to make the implementation easy.

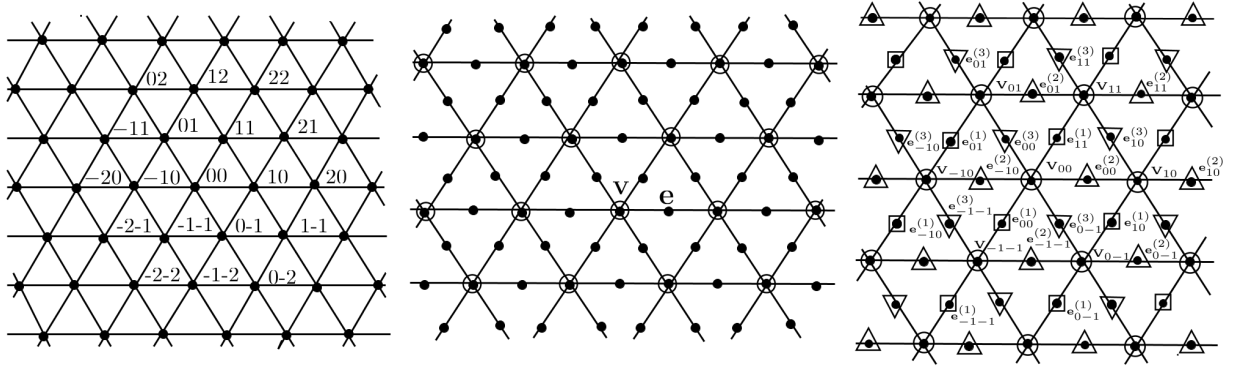


Figure 5.5: Left: Indices for nodes and initial data; Middle: Coarse mesh; Right: Initial data separated into 4 groups (vertex and edge nodes): $\{\mathbf{v}_k\}$, $\{\mathbf{e}_k^{(1)}\}$, $\{\mathbf{e}_k^{(2)}\}$, $\{\mathbf{e}_k^{(3)}\}$

We can represent the multiresolution algorithms for regular vertices as templates. Then, we can construct template-based multiwavelets. To do this approach, let \mathcal{M}_0 denotes the regular triangular mesh which represented from a regular infinite mesh $\mathcal{C} = \{\mathbf{c}_k\}_{k \in \mathbb{Z}^2}$ that can be given by a regular triangular mesh \mathcal{M}_0 . Here we separate the nodes of \mathcal{M}_0 , which is the regular triangular mesh, into two different groups. The first group containing the nodes with indices $(2k_1, 2k_2)$ of $2\mathbb{Z}^2$ for the coarse mesh and the remaining nodes with indices of

$\mathbf{Z}^2 \setminus (2\mathbf{Z}^2)$ will be in the second group.

See Fig. 5.5, on the left, we show the regular triangular mesh \mathcal{M}_0 , with indices of \mathbf{Z}^2 , which represents the regular infinite mesh $\mathcal{C} = \{\mathbf{c}_\mathbf{k}\}_{\mathbf{k} \in \mathbf{Z}^2}$. The middle of the figure shows the refinement (dyadic) and the nodes with \bigcirc form the coarse mesh. On the right of the figure, the circle \bigcirc denotes the first group type V **nodes** (or **vertex nodes**) and we make type E **nodes** (or **edge nodes**) as three groups with labels in:

$$\{2\mathbf{k} - (1, 1)\}_{\mathbf{k} \in \mathbf{Z}^2}, \{2\mathbf{k} + (1, 0)\}_{\mathbf{k} \in \mathbf{Z}^2}, \{2\mathbf{k} + (0, 1)\}_{\mathbf{k} \in \mathbf{Z}^2}. \quad (5.9)$$

which are denoted by \square , \triangle , and ∇ respectively.

For a regular mesh $\mathcal{C} = \{\mathbf{c}_\mathbf{k}\}_{\mathbf{k} \in \mathbf{Z}^2}$ with vertices $\mathbf{c}_\mathbf{k}$. Let $\{\mathbf{c}_\mathbf{k}^1\}_\mathbf{k}$ be the “approximation” and $\{\mathbf{d}_\mathbf{k}^{(1,\ell)}\}_\mathbf{k}$, $1 \leq \ell \leq 3$ be the “the details”. Associating the set of data $\{\mathbf{c}_{2\mathbf{k}}\}_{\mathbf{k} \in \mathbf{Z}^2}$ with type V nodes, and the three sets of the data $\{\mathbf{c}_{2\mathbf{k}-(1,1)}\}_{\mathbf{k} \in \mathbf{Z}^2}$, $\{\mathbf{c}_{2\mathbf{k}+(1,0)}\}_{\mathbf{k} \in \mathbf{Z}^2}$ and $\{\mathbf{c}_{2\mathbf{k}+(0,1)}\}_{\mathbf{k} \in \mathbf{Z}^2}$ associated with the previous three groups of type E nodes. Then, the decomposition algorithms and reconstruction algorithms can be given by templates. For initial data $\{\mathbf{c}_\mathbf{k}\}_\mathbf{k}$, denotes

$$\mathbf{v}_\mathbf{k} = \mathbf{c}_{2\mathbf{k}}, \mathbf{e}_\mathbf{k}^{(1)} = \mathbf{c}_{2\mathbf{k}-(1,1)}, \mathbf{e}_\mathbf{k}^{(2)} = \mathbf{c}_{2\mathbf{k}+(1,0)}, \mathbf{e}_\mathbf{k}^{(3)} = \mathbf{c}_{2\mathbf{k}+(0,1)}, \mathbf{k} \in \mathbf{Z}^2. \quad (5.10)$$

Now we have four groups of data. For $\mathbf{v}_\mathbf{k}$, we call type V vertices. For any of $\mathbf{e}_\mathbf{k}^{(\ell)}$, $\ell = 1, 2, 3$, we call type E vertices. Denote

$$\tilde{\mathbf{v}}_\mathbf{k} = \mathbf{c}_\mathbf{k}^1, \tilde{\mathbf{e}}_\mathbf{k}^{(1)} = \mathbf{d}_\mathbf{k}^{(1,1)}, \tilde{\mathbf{e}}_\mathbf{k}^{(2)} = \mathbf{d}_\mathbf{k}^{(1,2)}, \tilde{\mathbf{e}}_\mathbf{k}^{(3)} = \mathbf{d}_\mathbf{k}^{(1,3)}, \quad (5.11)$$

where $\mathbf{v}_\mathbf{k}$ and $\mathbf{e}_\mathbf{k}^{(\ell)}$, $\ell = 1, 2, 3$ are 1×2 row-vectors.

When we use the multifilter bank $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}\}$ as the analysis multifilter bank, the multiresolution decomposition algorithm for input data $\{\mathbf{c}_\mathbf{k}\}$ is

$$\begin{cases} \mathbf{c}_\mathbf{k}^1 = \frac{1}{4} \sum_{\mathbf{k}' \in \mathbf{Z}^2} \mathbf{c}_{\mathbf{k}'} \mathbf{P}_{\mathbf{k}'-2\mathbf{k}}^T, & \mathbf{d}_\mathbf{k}^{(1,1)} = \frac{1}{4} \sum_{\mathbf{k}' \in \mathbf{Z}^2} \mathbf{c}_{\mathbf{k}'} \mathbf{Q}_{\mathbf{k}'-2\mathbf{k}}^{(1)T}, \\ \mathbf{d}_\mathbf{k}^{(1,2)} = \frac{1}{4} \sum_{\mathbf{k}' \in \mathbf{Z}^2} \mathbf{c}_{\mathbf{k}'} \mathbf{Q}_{\mathbf{k}'-2\mathbf{k}}^{(2)T}, & \mathbf{d}_\mathbf{k}^{(1,3)} = \frac{1}{4} \sum_{\mathbf{k}' \in \mathbf{Z}^2} \mathbf{c}_{\mathbf{k}'} \mathbf{Q}_{\mathbf{k}'-2\mathbf{k}}^{(3)T}, \end{cases} \quad (5.12)$$

for $\mathbf{k}, \mathbf{k}' \in \mathbf{Z}^2$, where $\mathbf{c}_\mathbf{k}^1$, $\mathbf{d}_\mathbf{k}^{(1,1)}$, $\mathbf{d}_\mathbf{k}^{(1,2)}$ and $\mathbf{d}_\mathbf{k}^{(1,3)}$ are 1×2 “row-vectors” and the masks $\mathbf{P}_{\mathbf{k}'-2\mathbf{k}}$ and $\mathbf{Q}_{\mathbf{k}'-2\mathbf{k}}^{(\ell)}$, $\ell = 1, \dots, 3$, are 2×2 square matrices. If the synthesis multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}\}$ is biorthogonal to $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}\}$, then the input data $\{\mathbf{c}_\mathbf{k}\}$ can be recovered from $\mathbf{c}_\mathbf{k}^1$, $\mathbf{d}_\mathbf{k}^{(1,1)}$, $\mathbf{d}_\mathbf{k}^{(1,2)}$ and $\mathbf{d}_\mathbf{k}^{(1,3)}$ by the multiresolution reconstruction algorithm:

$$\mathbf{c}'_\mathbf{k} = \sum_{\mathbf{k}' \in \mathbf{Z}^2} \{\mathbf{c}_{\mathbf{k}'}^1 \tilde{\mathbf{P}}_{\mathbf{k}-2\mathbf{k}'} + \mathbf{d}_{\mathbf{k}'}^{(1,1)} \tilde{\mathbf{Q}}_{\mathbf{k}-2\mathbf{k}'}^{(1)} + \mathbf{d}_{\mathbf{k}'}^{(1,2)} \tilde{\mathbf{Q}}_{\mathbf{k}-2\mathbf{k}'}^{(2)} + \mathbf{d}_{\mathbf{k}'}^{(1,3)} \tilde{\mathbf{Q}}_{\mathbf{k}-2\mathbf{k}'}^{(3)}\}, \quad (5.13)$$

where $\mathbf{k} \in \mathbf{Z}^2$ and the multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}\}$ is the synthesis multifilter bank. $\{\tilde{\mathbf{v}}_\mathbf{k}\}_\mathbf{k}$ is called the lowpass output and $\{\mathbf{d}_\mathbf{k}^{(1,1)}\}_\mathbf{k}$, $\{\mathbf{d}_\mathbf{k}^{(1,2)}\}_\mathbf{k}$, $\{\mathbf{d}_\mathbf{k}^{(1,3)}\}_\mathbf{k}$ are called the highpass outputs of $\{\mathbf{c}_\mathbf{k}\}_\mathbf{k}$ respectively.

Theorem 3. *If $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}\}$ are biorthogonal multifilter banks, then the input data $\{\mathbf{c}_\mathbf{k}\}$ can be recovered from its approximation $\{\mathbf{c}_\mathbf{k}^1\}$ and details $\{\mathbf{d}_\mathbf{k}^{(1,1)}\}$, $\{\mathbf{d}_\mathbf{k}^{(1,2)}\}$, $\{\mathbf{d}_\mathbf{k}^{(1,3)}\}$, namely the input data $\{\mathbf{c}_\mathbf{k}\}$ is exactly $\{\mathbf{c}'_\mathbf{k}\}$.*

Proof: Let $\mathbf{c}^1(2\boldsymbol{\omega})$, $\mathbf{d}^{(1,1)}(2\boldsymbol{\omega})$, $\mathbf{d}^{(1,2)}(2\boldsymbol{\omega})$, $\mathbf{d}^{(1,3)}(2\boldsymbol{\omega})$, $\mathbf{c}(\boldsymbol{\omega})$, and $\mathbf{c}'(\boldsymbol{\omega})$ denote the Z-transforms of $\mathbf{c}_{\mathbf{k}}^1$, $\mathbf{d}_{\mathbf{k}}^{(1,1)}$, $\mathbf{d}_{\mathbf{k}}^{(1,2)}$, $\mathbf{d}_{\mathbf{k}}^{(1,3)}$, $\mathbf{c}_{\mathbf{k}}$, and $\mathbf{c}'_{\mathbf{k}}$ respectively, where $\mathbf{k} \in \mathbf{Z}^2$. The Z-transforms $\mathbf{c}(\boldsymbol{\omega})$ of $\mathbf{c}_{\mathbf{k}} = \{\mathbf{c}_{\mathbf{k}}\}_{\mathbf{k}}$ is $\mathbf{c}(\boldsymbol{\omega}) = \frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^2} \mathbf{c}_{\mathbf{k}} e^{-i\mathbf{k}\boldsymbol{\omega}}$. In the frequency domain, we have

$$\mathbf{c}^1(2\boldsymbol{\omega}) = \frac{1}{4} \sum_{j=0}^3 \mathbf{c}(\boldsymbol{\omega} + \pi\boldsymbol{\eta}_j) \overline{\mathbf{P}(\boldsymbol{\omega} + \pi\boldsymbol{\eta}_j)}^T,$$

$$\mathbf{d}^{(1,1)}(2\boldsymbol{\omega}) = \frac{1}{4} \sum_{j=0}^3 \mathbf{c}(\boldsymbol{\omega} + \pi\boldsymbol{\eta}_j) \overline{\mathbf{Q}^{(1)}(\boldsymbol{\omega} + \pi\boldsymbol{\eta}_j)}^T,$$

$$\mathbf{d}^{(1,2)}(2\boldsymbol{\omega}) = \frac{1}{4} \sum_{j=0}^3 \mathbf{c}(\boldsymbol{\omega} + \pi\boldsymbol{\eta}_j) \overline{\mathbf{Q}^{(2)}(\boldsymbol{\omega} + \pi\boldsymbol{\eta}_j)}^T,$$

$$\mathbf{d}^{(1,3)}(2\boldsymbol{\omega}) = \frac{1}{4} \sum_{j=0}^3 \mathbf{c}(\boldsymbol{\omega} + \pi\boldsymbol{\eta}_j) \overline{\mathbf{Q}^{(3)}(\boldsymbol{\omega} + \pi\boldsymbol{\eta}_j)}^T,$$

and

$$\begin{aligned} \mathbf{c}'(\boldsymbol{\omega}) = & 4 \left(\mathbf{c}^1(2\boldsymbol{\omega}) \tilde{\mathbf{P}}(\boldsymbol{\omega}) + \mathbf{d}^{(1,1)}(2\boldsymbol{\omega}) \tilde{\mathbf{Q}}^{(1)}(\boldsymbol{\omega}) + \mathbf{d}^{(1,2)}(2\boldsymbol{\omega}) \tilde{\mathbf{Q}}^{(2)}(\boldsymbol{\omega}) \right. \\ & \left. + \mathbf{d}^{(1,3)}(2\boldsymbol{\omega}) \tilde{\mathbf{Q}}^{(3)}(\boldsymbol{\omega}) \right). \end{aligned}$$

Suppose the multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}\}$ are biorthogonal. By plugging $\mathbf{c}^1(2\boldsymbol{\omega})$, $\mathbf{d}^{(1,1)}(2\boldsymbol{\omega})$, $\mathbf{d}^{(1,2)}(2\boldsymbol{\omega})$, $\mathbf{d}^{(1,3)}(2\boldsymbol{\omega})$ into $\mathbf{c}'(\boldsymbol{\omega})$ and if the conditions (5.2) hold. Then, $\mathbf{c}'(\boldsymbol{\omega}) = \mathbf{c}(\boldsymbol{\omega})$. Hence, the biorthogonality of $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}\}$ implies that the original data can be recovered from the lowpass and highpass outputs by the multiwavelet reconstruction algorithms.

As in the paper [76], we consider $\mathbf{c}_{\mathbf{k}}$ with \mathbf{k} in four different cases: $(2j_1, 2j_2)$, $(2j_1 - 1, 2j_2 - 1)$, $(2j_1 + 1, 2j_2)$, $(2j_1, 2j_2 + 1)$, and with the definitions of $\mathbf{v}_{\mathbf{k}}, \mathbf{e}_{\mathbf{k}}^{(\ell)}$, $\ell = 1, 2, 3$, we

can write the above reconstruction algorithm as

$$\left\{ \begin{aligned} \mathbf{v}_k &= \sum_{n \in \mathbb{Z}^2} \{ \tilde{\mathbf{v}}_{k-n} \tilde{\mathbf{P}}_{2n} + \tilde{\mathbf{e}}_{k-n}^{(1)} \tilde{\mathbf{Q}}_{2n}^{(1)} + \tilde{\mathbf{e}}_{k-n}^{(2)} \tilde{\mathbf{Q}}_{2n}^{(2)} + \tilde{\mathbf{e}}_{k-n}^{(3)} \tilde{\mathbf{Q}}_{2n}^{(3)} \}, \\ \mathbf{e}_k^{(1)} &= \sum_{n \in \mathbb{Z}^2} \{ \tilde{\mathbf{v}}_{k-n} \tilde{\mathbf{P}}_{2n-(1,1)} + \tilde{\mathbf{e}}_{k-n}^{(1)} \tilde{\mathbf{Q}}_{2n-(1,1)}^{(1)} + \tilde{\mathbf{e}}_{k-n}^{(2)} \tilde{\mathbf{Q}}_{2n-(1,1)}^{(2)} \\ &\quad + \tilde{\mathbf{e}}_{k-n}^{(3)} \tilde{\mathbf{Q}}_{2n-(1,1)}^{(3)} \}, \\ \mathbf{e}_k^{(2)} &= \sum_{n \in \mathbb{Z}^2} \{ \tilde{\mathbf{v}}_{k-n} \tilde{\mathbf{P}}_{2n+(1,0)} + \tilde{\mathbf{e}}_{k-n}^{(1)} \tilde{\mathbf{Q}}_{2n+(1,0)}^{(1)} + \tilde{\mathbf{e}}_{k-n}^{(2)} \tilde{\mathbf{Q}}_{2n+(1,0)}^{(2)} \\ &\quad + \tilde{\mathbf{e}}_{k-n}^{(3)} \tilde{\mathbf{Q}}_{2n+(1,0)}^{(3)} \}, \\ \mathbf{e}_k^{(3)} &= \sum_{n \in \mathbb{Z}^2} \{ \tilde{\mathbf{v}}_{k-n} \tilde{\mathbf{P}}_{2n+(0,1)} + \tilde{\mathbf{e}}_{k-n}^{(1)} \tilde{\mathbf{Q}}_{2n+(0,1)}^{(1)} + \tilde{\mathbf{e}}_{k-n}^{(2)} \tilde{\mathbf{Q}}_{2n+(0,1)}^{(2)} \\ &\quad + \tilde{\mathbf{e}}_{k-n}^{(3)} \tilde{\mathbf{Q}}_{2n+(0,1)}^{(3)} \}. \end{aligned} \right. \quad (5.14)$$

As in [76], we observe that when we set the “details“ $\tilde{\mathbf{e}}_k^{(\ell)}$, $\ell = 1, 2, 3$ to be 1×2 zero row vectors, the reconstruction algorithm (5.14) reduced to the subdivision algorithm:

$$\left\{ \begin{aligned} \mathbf{v}_k &= \sum_{n \in \mathbb{Z}^2} \tilde{\mathbf{v}}_{k-n} \tilde{\mathbf{P}}_{2n} \\ \mathbf{e}_k^{(1)} &= \sum_{n \in \mathbb{Z}^2} \tilde{\mathbf{v}}_{k-n} \tilde{\mathbf{P}}_{2n-(1,1)} \\ \mathbf{e}_k^{(2)} &= \sum_{n \in \mathbb{Z}^2} \tilde{\mathbf{v}}_{k-n} \tilde{\mathbf{P}}_{2n+(1,0)} \\ \mathbf{e}_k^{(3)} &= \sum_{n \in \mathbb{Z}^2} \tilde{\mathbf{v}}_{k-n} \tilde{\mathbf{P}}_{2n+(0,1)}, \end{aligned} \right. \quad (5.15)$$

from the above algorithm we can derive the subdivision templates.

The algorithms are given by templates by associating the outputs $\tilde{\mathbf{v}}_k, \tilde{\mathbf{e}}_k^{(\ell)}$, $\ell = 1, 2, 3$ with the node of \mathcal{M}_0 , which is the regular triangular mesh. So, we will associate the lowpass output $\{\tilde{\mathbf{v}}_k\}_{k \in \mathbb{Z}^2}$ with type V nodes $(2k_1, 2k_2)$ and associate the highpass outputs $\tilde{\mathbf{e}}_k^{(\ell)}$, $\ell = 1, 2, 3$ with the type E nodes $(k_1 - 1, k_2 - 1)$, $(k_1 + 1, k_2)$ and $(k_1, k_2 + 1)$, respectively.

Next, we will show two-dimensional biorthogonal multiwavelets and associated multiresolution algorithm templates.

2-Step Multiwavelet Multiresolution Algorithm

We will present a 2-step multiwavelet algorithm. Let 1×2 row vectors $\{\mathbf{v}\}$ and $\{\mathbf{e}\}$ (or a given triangular mesh C as introduced in the beginning of this section), Fig. (5.6) and (5.7) shows the multiresolution decomposition algorithm in the formulas (5.16)-(5.17), the

matrices \mathbf{B} , \mathbf{M} , \mathbf{A} , \mathbf{C} , are 2×2 matrices and the entries of these matrices are some constants in \mathbb{R} to be determined.

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

2-step Multiwavelet Decomposition Algorithm:

$$\text{Step 1. } \tilde{\mathbf{v}} = \{\mathbf{v} - (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) \mathbf{M}\} \mathbf{B}^{-1}; \quad (5.16)$$

$$\text{Step 2. } \tilde{\mathbf{e}} = \mathbf{e} - (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1) \mathbf{A} - (\tilde{\mathbf{v}}_2 + \tilde{\mathbf{v}}_3) \mathbf{C}. \quad (5.17)$$

By the formulas given in (5.16), we can obtain $\tilde{\mathbf{v}}$, which is associated with the vertex nodes

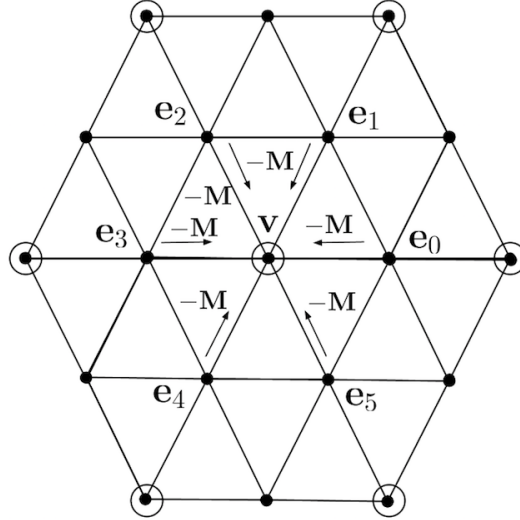


Figure 5.6: Decomposition Alg. Step 1 (template to get lowpass output \mathbf{v} .
)

(type V nodes) of \mathcal{M}_0 . Then, in Step 2, which is given in (5.17), we use the obtained $\tilde{\mathbf{v}}$ to obtain other 3 highpass outputs $\{\tilde{\mathbf{e}}\} (= \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(1)}\} \cup \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(2)}\} \cup \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(3)}\})$ that are associated with the edge nodes of \mathcal{M}_0 .

In Fig. 5.8, we show the synthesis algorithm and the analysis algorithm are by (5.18)-(5.19), we give the multiresolution reconstruction algorithm, the matrices \mathbf{B} , \mathbf{M} , \mathbf{A} , \mathbf{C} are the same 2×2 matrices used in the decomposition algorithm.

2-step Multiwavelet Reconstruction Algorithm:

$$\text{Step 1. } \mathbf{e} = \tilde{\mathbf{e}} + (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1) \mathbf{A} + (\tilde{\mathbf{v}}_2 + \tilde{\mathbf{v}}_3) \mathbf{C}; \quad (5.18)$$

$$\text{Step 2. } \mathbf{v} = \tilde{\mathbf{v}} \mathbf{B} + (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) \mathbf{M}. \quad (5.19)$$

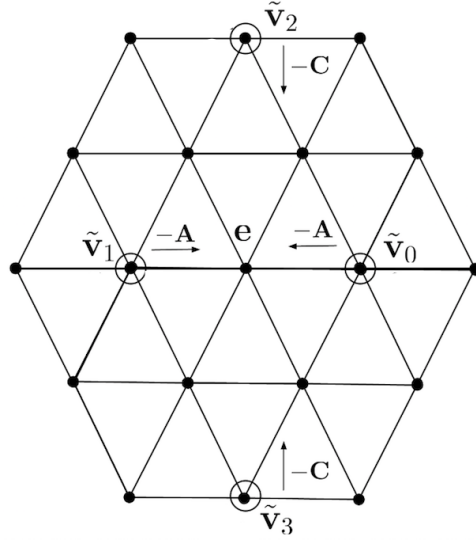


Figure 5.7: Decomposition Alg. Step 2.

First in Step 1, we put $\tilde{\mathbf{e}}$ instead of \mathbf{e} , in (5.18). Then in Step 2 with \mathbf{e} that is found in

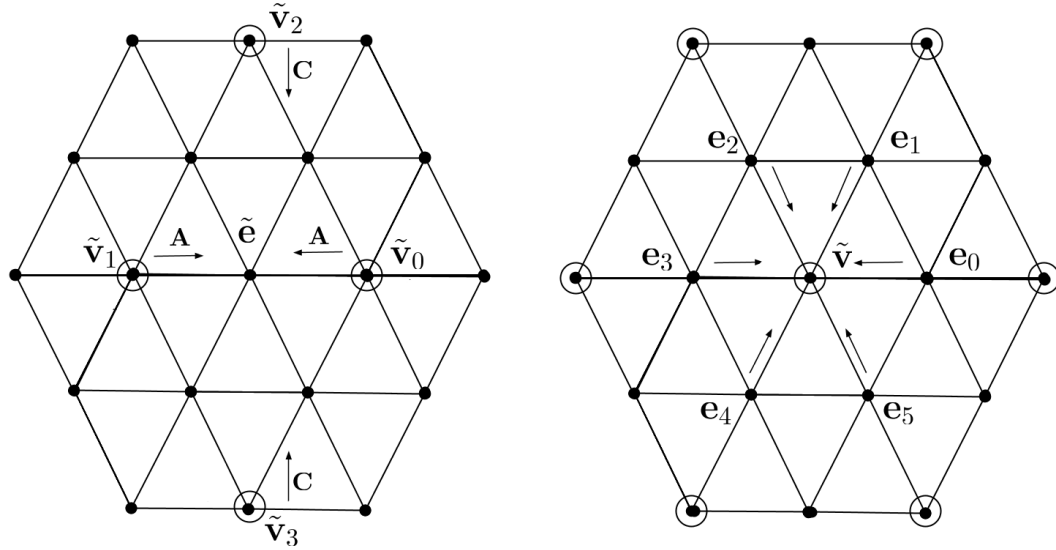


Figure 5.8: 2-D 2-step multiwavelet reconstruction algorithm. Left: Step 1; Right: Step 2.

Step 1, we replace $\tilde{\mathbf{v}}$ by \mathbf{v} that is given in (5.19). So, the inputs here are $\tilde{\mathbf{v}}$, $\tilde{\mathbf{e}}$ and the outputs are \mathbf{v} , \mathbf{e} . We derive the analysis lowpass multifilter \mathbf{P}_{k_1, k_2} which is satisfy the sum rule of order 1, namely:

$$\begin{aligned} \mathbf{P}_{0,0} &= 4 (\mathbf{B}^{-1})^T, \\ \mathbf{P}_{1,0} &= \mathbf{P}_{0,1} = \mathbf{P}_{0,-1} = \mathbf{P}_{-1,0} = \mathbf{P}_{-1,-1} = \mathbf{P}_{1,1} = -4 (\mathbf{B}^{-1})^T \mathbf{M}^T. \end{aligned} \quad (5.20)$$

and we derive the synthesis lowpass multifilter $\tilde{\mathbf{P}}_{k_1, k_2}$ which is satisfy the sum rule of order 1, namely:

$$\begin{aligned}\tilde{\mathbf{P}}_{0,0} &= \mathbf{B} + 6 \mathbf{A}\mathbf{M}, \\ \tilde{\mathbf{P}}_{1,0} &= \tilde{\mathbf{P}}_{0,1} = \tilde{\mathbf{P}}_{0,-1} = \tilde{\mathbf{P}}_{-1,0} = \tilde{\mathbf{P}}_{-1,-1} = \tilde{\mathbf{P}}_{1,1} = \mathbf{A}, \\ \tilde{\mathbf{P}}_{2,0} &= \tilde{\mathbf{P}}_{0,2} = \tilde{\mathbf{P}}_{0,-2} = \tilde{\mathbf{P}}_{-2,0} = \tilde{\mathbf{P}}_{-2,-2} = \tilde{\mathbf{P}}_{2,2} = \mathbf{A}\mathbf{M} + 2 \mathbf{C}^T\mathbf{M}, \\ \tilde{\mathbf{P}}_{1,2} &= \tilde{\mathbf{P}}_{2,1} = \tilde{\mathbf{P}}_{-1,1} = \tilde{\mathbf{P}}_{1,-1} = \tilde{\mathbf{P}}_{-1,-2} = \tilde{\mathbf{P}}_{-2,-1} = \mathbf{C}.\end{aligned}$$

Denote

$$\left\{ \begin{aligned} \mathbf{F}_1(\boldsymbol{\omega}) &= \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -(1 + \frac{1}{xy})\mathbf{A}^T - (\frac{1}{x} + \frac{1}{y})\mathbf{C}^T & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -(1 + x)\mathbf{A}^T - (xy + \frac{1}{y})\mathbf{C}^T & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ -(1 + y)\mathbf{A}^T - (xy + \frac{1}{x})\mathbf{C}^T & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{8 \times 8}, \\ \mathbf{F}_0(\boldsymbol{\omega}) &= \begin{bmatrix} (\mathbf{B}^{-1})^T & -(1 + xy)(\mathbf{B}^{-1})^T\mathbf{M}^T & -(1 + \frac{1}{x})(\mathbf{B}^{-1})^T\mathbf{M}^T & -(1 + \frac{1}{y})(\mathbf{B}^{-1})^T\mathbf{M}^T \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{8 \times 8}. \end{aligned} \right. \quad (5.21)$$

and the inverse of $\mathbf{F}(\boldsymbol{\omega})$ is a matrix. The entries of $\mathbf{F}(\boldsymbol{\omega})$ are also polynomials of x, y . More precisely, $\tilde{\mathbf{F}}_1(\boldsymbol{\omega}) = (\mathbf{F}_1(\boldsymbol{\omega})^{-1})^*$ and $\tilde{\mathbf{F}}_0(\boldsymbol{\omega}) = (\mathbf{F}_0(\boldsymbol{\omega})^{-1})^*$ are given by

$$\left\{ \begin{aligned} \tilde{\mathbf{F}}_1(\boldsymbol{\omega}) &= \begin{bmatrix} \mathbf{I}_2 & (1 + xy)\mathbf{A} + (x + y)\mathbf{C} & (1 + \frac{1}{x})\mathbf{A} + (\frac{1}{xy} + y)\mathbf{C} & (1 + \frac{1}{y})\mathbf{A} + (\frac{1}{xy} + x)\mathbf{C} \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{8 \times 8}, \\ \tilde{\mathbf{F}}_0(\boldsymbol{\omega}) &= \begin{bmatrix} \mathbf{B} & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1 + \frac{1}{xy})\mathbf{M} & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1 + x)\mathbf{M} & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ (1 + y)\mathbf{M} & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{8 \times 8}. \end{aligned} \right. \quad (5.22)$$

The analysis multifilter bank $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}\}$ and the synthesis multifilter bank $\{\tilde{\mathbf{P}}, \}$ can be written as the following:

$$[\mathbf{P}(\omega), \mathbf{Q}^{(1)}(\omega), \mathbf{Q}^{(2)}(\omega), \mathbf{Q}^{(3)}(\omega)]^T = \mathbf{F}_1(2\omega)\mathbf{F}_0(2\omega)\mathbf{I}_0(\omega), \quad (5.23)$$

$$[\tilde{\mathbf{P}}(\omega), \tilde{\mathbf{Q}}^{(1)}(\omega), \tilde{\mathbf{Q}}^{(2)}(\omega), \tilde{\mathbf{Q}}^{(3)}(\omega)]^T = \frac{1}{4}\tilde{\mathbf{F}}_1(2\omega)\tilde{\mathbf{F}}_0(2\omega)\mathbf{I}_0(\omega), \quad (5.24)$$

where $\mathbf{I}_0(\omega)$ is defined by (5.3).

By solving the system of equations for a sum rule order one both of $\tilde{\mathbf{P}}$ and \mathbf{P} , and for a vanishing moment order one of $\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}$, we have 8 free parameters which are $b_{22}, m_{12}, m_{22}, c_{11}, c_{22}, c_{12}, c_{21}, a_{22}$.

$$\begin{aligned} a_{11} &= -c_{11} + \frac{1}{2}, \quad a_{12} = -c_{12}, \quad a_{21} = -c_{21}, \quad b_{11} = 4, \quad b_{12} = -6m_{12}, \quad b_{21} = 0, \\ m_{11} &= -\frac{1}{2}, \quad m_{21} = 0. \end{aligned}$$

if we choose

$$b_{22} = \frac{3028}{4643}, \quad m_{12} = \frac{1}{3273}, \quad m_{22} = \frac{49}{759}, \quad c_{11} = -\frac{1}{4}, \quad c_{12} = -\frac{51}{149}, \quad c_{21} = 0, \quad a_{22} = \frac{177}{467}, \quad c_{22} = -\frac{131}{715}.$$

By using the smoothness formula, we can obtain $\tilde{\Phi} \in W^{0.4408}$. The mask \mathbf{P} is not in $L^2(\mathbb{R}^2)^r$. If we choose $b_{22} = \frac{97}{39}$, $m_{12} = \frac{5419}{20}$, $m_{22} = \frac{42}{167}$, then \mathbf{P} is in $L^2(\mathbb{R}^2)^r$ and we can obtain $\Phi \in W^{0.1379}$ is supported on $[-1, 1]^2$, but $\tilde{\Phi}$ is not in $L^2(\mathbb{R}^2)^2$. If we change the values of the parameters

$$\begin{aligned} b_{22} &= \frac{2184}{859}, \quad m_{12} = \frac{-155}{377}, \quad m_{22} = \frac{111}{290}, \quad c_{11} = -\frac{53}{185}, \quad c_{12} = \frac{243}{1486}, \quad c_{21} = \frac{-83}{638}, \quad a_{22} = \frac{-103}{335}, \\ c_{22} &= \frac{43}{801}. \end{aligned}$$

then, $\Phi \in W^{0.0074}$ and $\tilde{\Phi} \in W^{0.4378}$ are in $L^2(\mathbb{R}^2)^2$. We can choose another values of the parameters such that Φ is in $L^2(\mathbb{R}^2)^2$, but we cannot get smoother $\tilde{\Phi}$ as long as its dual is in L^2 .

The lowpass analysis mask \mathbf{P} and the lowpass synthesis mask $\tilde{\mathbf{P}}$ will be as follows:

$$\mathbf{P}(\omega) = \begin{bmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned}
P_{11}(\omega) &= (0.1249994700x^2y^2 + 0.1249994700x^2y + 0.1249994700xy^2 + 0.2499989400xy \\
&\quad + 0.1249994700x + 0.1249994700y + 0.1249994700)/(xy), \\
P_{12}(\omega) &= (0.1249994700x^2y^2 + 0.1249994700x^2y + 0.1249994700xy^2 + 0.2499989400xy \\
&\quad + 0.1249994700x + 0.1249994700y + 0.1249994700)/(xy), \\
P_{21}(\omega) &= (0.04042755720x^2y^2 + 0.04042755720x^2y + 0.04042755720xy^2 \\
&\quad - .2425604268xy + 0.0404275572x + 0.0404275572y + 0.04042755720)/(xy), \\
P_{22}(\omega) &= (-0.1505426262x^2y^2 - 0.1505426262x^2y - 0.1505426262xy^2 + 0.393312000xy \\
&\quad - 0.1505426262x - 0.1505426262y - 0.3827562500)/(xy).
\end{aligned}$$

and

$$\tilde{\mathbf{P}}(\omega) = \begin{bmatrix} \tilde{P}_{11}(\omega) & \tilde{P}_{12}(\omega) \\ \tilde{P}_{21}(\omega) & \tilde{P}_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned}
\tilde{P}_{11}(\omega) &= (-0.02669x^4y^4 - 0.0716225x^4y^3 - 0.02669x^4y^2 - 0.0716225x^3y^4 \\
&\quad + 0.1966225x^3y^3 + 0.1966225x^3y^2 - 0.0716225x^3y - 0.02669x^2y^4 \\
&\quad + 0.1966225x^2y^3 + 0.410125x^2y^2 + 0.1966225x^2y - 0.02669x^2 \\
&\quad - 0.0716225xy^3 + 0.1966222xy^2 + 0.1966225xy - 0.0716225x \\
&\quad - 0.02669y^2 - 0.0716225y - 0.02669)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_{12}(\omega) &= (-0.00629825x^4y^4 + 0.0408825x^4y^3 + 0.0408825x^3y^4 - 0.00629825x^4y^2 \\
&\quad - 0.0408825x^3y^3 - 0.00629825x^2y^4 - 0.0408825x^3y^2 - 0.0408825x^2y^3 \\
&\quad + 0.0408825x^3y + 0.0377900x^2y^2 + 0.0408825xy^3 - 0.0408825x^2y \\
&\quad - 0.0408825xy^2 - 0.00629825x^2 - 0.0408825xy - 0.00629825y^2 \\
&\quad + 0.0408825x + 0.0408825y - 0.00629825)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_{21}(\omega) &= (0.25(0.065047x^4y^4 - 0.13009x^4y^3 - 0.13009x^3y^4 + 0.065047x^4y^2 \\
&\quad + 0.13009x^3y^3 + 0.065047x^2y^4 + 0.13009x^3y^2 + 0.13009x^2y^3 \\
&\quad - 0.13009x^3y - 0.39028x^2y^2 - 0.13009xy^3 + 0.13009x^2y \\
&\quad + 0.13009xy^2 + 0.065047x^2 + 0.13009xy + 0.065047y^2 \\
&\quad - 0.13009x - 0.13009y + 0.065047))/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_{22}(\omega) &= (-0.00577550x^4y^4 + 0.01342075x^4y^3 + 0.01342075x^3y^4 - 0.00577550x^4y^2 \\
&\quad - 0.0768650x^3y^3 - 0.00577550x^2y^4 - 0.0768650x^3y^2 - 0.0768650x^2y^3 \\
&\quad + 0.01342075x^3y + 0.378875x^2y^2 + 0.01342075xy^3 - 0.0768650x^2y \\
&\quad - 0.0768650xy^2 - 0.00577550x^2 - 0.0768650xy - 0.00577550y^2 \\
&\quad + 0.01342075x + 0.01342075y - 0.00577550)/(x^2y^2).
\end{aligned}$$

$$\mathbf{Q}^{(1)}(\omega) = \begin{bmatrix} Q_{11}^{(1)}(\omega) & Q_{12}^{(1)}(\omega) \\ Q_{21}^{(1)}(\omega) & Q_{22}^{(1)}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} Q_{11}^{(1)}(\omega) = & -(0.1035699357x^4y^4 + 0.1035699357x^4y^3 + 0.1035699357x^3y^4 - 0.04107005672x^4y^2 \\ & + 0.1650648923x^3y^3 - 0.04107005672x^2y^4 - 0.04107005672x^4y + 0.06249987898x^3y^2 \\ & + 0.06249987898x^2y^3 - 0.04107005672xy^4 - 0.04006611763x^3y - 0.7928660586x^2y^2 \\ & - 0.04006611763xy^3 - 0.04107005672x^3 + 0.06249987898x^2y + 0.06249987898xy^2 \\ & - 0.04107005672y^3 - 0.04107005672x^2 + 0.1650648923xy - 0.04107005672y^2 \\ & + 0.1035699357x + 0.1035699357y + 0.1035699357)/(x^3y^3), \end{aligned}$$

$$\begin{aligned} Q_{12}^{(1)}(\omega) = & -(-0.01958517022x^4y^4 - 0.01958517022x^4y^3 - 0.01958517022x^3y^4 + 0.01958517022x^4y^2 \\ & + 0.05116844502x^3y^3 + 0.01958517022x^2y^4 + 0.01958517022x^4y + 0.01958517022xy^4 \\ & - 0.05116844502x^3y - 0.03917034044x^2y^2 - 0.05116844502xy^3 + 0.01958517022x^3 \\ & + 0.01958517022y^3 + 0.01958517022x^2 + 0.05116844502xy + 0.01958517022y^2 \\ & - 0.01958517022x - 0.01958517022y - 0.01958517022)/(x^3y^3), \end{aligned}$$

$$\begin{aligned} Q_{21}^{(1)}(\omega) = & -(-0.03287040141x^4y^4 - 0.03287040141x^4y^3 - 0.03287040141x^3y^4 + 0.02261075355x^4y^2 \\ & + 0.03369734830x^3y^3 + 0.02261075355x^2y^4 + 0.02261075355x^4y - 0.01025964786x^3y^2 \\ & - 0.01025964786x^2y^3 + 0.02261075355xy^4 + 0.02786053886x^3y - 0.06574080282x^2y^2 \\ & + 0.02786053886xy^3 + 0.02261075355x^3 - 0.01025964786x^2y - 0.01025964786xy^2 \\ & + 0.02261075355y^3 + 0.02261075355x^2 + 0.03369734830xy + 0.02261075355y^2 \\ & - 0.03287040141x - 0.03287040141y - 0.03287040141)/(x^3y^3), \end{aligned}$$

$$\begin{aligned} Q_{22}^{(1)}(\omega) = & -(0.04628739346x^4y^4 + 0.04628739346x^4y^3 + 0.04628739346x^3y^4 - 0.8081660510x^4y^2 \\ & - 0.1209348371x^3y^3 - 0.8081660510x^2y^4 - 0.8081660510x^4y + 0.03820573295x^3y^2 \\ & + 0.03820573295x^2y^3 - 0.8081660510xy^4 + 0.02111418617x^3y - 0.9074291765x^2y^2 \\ & + 0.02111418617xy^3 - 0.8081660510x^3 + 0.03820573295x^2y + 0.03820573295xy^2 \\ & - 0.8081660510y^3 - 0.8081660510x^2 - 0.1209348371xy - 0.8081660510y^2 \\ & + 0.04628739346x + 0.04628739346y + 0.04628739346)/(x^3y^3). \end{aligned}$$

$$\mathbf{Q}^{(2)}(\omega) = \begin{bmatrix} Q_{11}^{(2)}(\omega) & Q_{12}^{(2)}(\omega) \\ Q_{21}^{(2)}(\omega) & Q_{22}^{(2)}(\omega) \end{bmatrix}$$

where

$$\begin{aligned}
Q_{11}^{(2)}(\omega) = & -(-0.04107005672x^4y^6 - 0.04107005672x^4y^5 - 0.04107005672x^3y^6 + 0.1035699357x^4y^4 \\
& - 0.04006611763x^3y^5 + 0.1035699357x^4y^3 + 0.06249987898x^3y^4 - 0.04107005672x^2y^5 \\
& + 0.1650648923x^3y^3 + 0.06249987898x^2y^4 + 0.1035699357x^3y^2 - 0.7928660586x^2y^3 \\
& + 0.1035699357xy^4 + 0.06249987898x^2y^2 + 0.1650648923xy^3 - 0.04107005672x^2y \\
& + 0.06249987898xy^2 + 0.1035699357y^3 - 0.04006611763xy + 0.1035699357y^2 \\
& - 0.04107005672x - 0.04107005672y - 0.04107005672)/(xy^3),
\end{aligned}$$

$$\begin{aligned}
Q_{12}^{(2)}(\omega) = & -(0.01958517022x^4y^6 + 0.01958517022x^4y^5 + 0.01958517022x^3y^6 - 0.01958517022x^4y^4 \\
& - 0.05116844502x^3y^5 - 0.01958517022x^4y^3 + 0.01958517022x^2y^5 + 0.05116844502x^3y^3 \\
& - 0.01958517022x^3y^2 - 0.03917034044x^2y^3 - 0.01958517022xy^4 + 0.05116844502xy^3 \\
& + 0.01958517022x^2y - 0.01958517022y^3 - 0.05116844502xy - 0.01958517022y^2 \\
& + 0.01958517022x + 0.01958517022y + 0.01958517022)/(xy^3),
\end{aligned}$$

$$\begin{aligned}
Q_{21}^{(2)}(\omega) = & -(0.02261075355x^4y^6 + 0.02261075355x^4y^5 + 0.02261075355x^3y^6 - 0.03287040141x^4y^4 \\
& + 0.02786053886x^3y^5 - 0.03287040141x^4y^3 - 0.01025964786x^3y^4 + 0.02261075355x^2y^5 \\
& + 0.03369734830x^3y^3 - 0.01025964786x^2y^4 - 0.03287040141x^3y^2 - 0.06574080282x^2y^3 \\
& - 0.03287040141xy^4 - 0.01025964786x^2y^2 + 0.03369734830xy^3 + 0.02261075355x^2y \\
& - 0.01025964786xy^2 - 0.03287040141y^3 + 0.02786053886xy - 0.03287040141y^2 \\
& + 0.02261075355x + 0.02261075355y + 0.02261075355)/(xy^3),
\end{aligned}$$

$$\begin{aligned}
Q_{22}^{(2)}(\omega) = & -(-0.8081660510x^4y^6 - 0.8081660510x^4y^5 - 0.8081660510x^3y^6 + 0.04628739346x^4y^4 \\
& + 0.02111418617x^3y^5 + 0.04628739346x^4y^3 + 0.03820573295x^3y^4 - 0.8081660510x^2y^5 \\
& - 0.1209348371x^3y^3 + 0.03820573295x^2y^4 + 0.04628739346x^3y^2 - 0.9074291765x^2y^3 \\
& + 0.04628739346xy^4 + 0.03820573295x^2y^2 - 0.1209348371xy^3 - 0.8081660510x^2y \\
& + 0.03820573295xy^2 + 0.04628739346y^3 + 0.02111418617xy + 0.04628739346y^2 \\
& - 0.8081660510x - 0.8081660510y - 0.8081660510)/(xy^3).
\end{aligned}$$

$$\mathbf{Q}^{(3)}(\omega) = \begin{bmatrix} Q_{11}^{(3)}(\omega) & Q_{12}^{(3)}(\omega) \\ Q_{21}^{(3)}(\omega) & Q_{22}^{(3)}(\omega) \end{bmatrix}$$

where

$$\begin{aligned}
Q_{11}^{(3)}(\omega) = & -(-0.04107005672x^6y^4 - 0.04107005672x^6y^3 - 0.04107005672x^5y^4 - 0.04006611763x^5y^3 \\
& + 0.1035699357x^4y^4 - 0.04107005672x^5y^2 + 0.06249987898x^4y^3 + 0.1035699357x^3y^4 \\
& + 0.06249987898x^4y^2 + 0.1650648923x^3y^3 + 0.1035699357x^4y - 0.7928660586x^3y^2 \\
& + 0.1035699357x^2y^3 + 0.1650648923x^3y + 0.06249987898x^2y^2 + 0.1035699357x^3 \\
& + 0.06249987898x^2y - 0.04107005672xy^2 + 0.1035699357x^2 - 0.04006611763xy \\
& - 0.04107005672x - 0.04107005672y - 0.04107005672)/(x^3y),
\end{aligned}$$

$$\begin{aligned}
Q_{12}^{(3)}(\omega) = & -(0.01958517022x^6y^4 + 0.01958517022x^6y^3 + 0.01958517022x^5y^4 - 0.05116844502x^5y^3 \\
& - 0.01958517022x^4y^4 + 0.01958517022x^5y^2 - 0.01958517022x^3y^4 + 0.05116844502x^3y^3 \\
& - 0.01958517022x^4y - 0.03917034044x^3y^2 - 0.01958517022x^2y^3 + 0.05116844502x^3y \\
& - 0.01958517022x^3 + 0.01958517022xy^2 - 0.01958517022x^2 - 0.05116844502xy \\
& + 0.01958517022x + 0.01958517022y + 0.01958517022)/(x^3y),
\end{aligned}$$

$$\begin{aligned}
Q_{21}^{(3)}(\omega) = & -(0.02261075355x^6y^4 + 0.02261075355x^6y^3 + 0.02261075355x^5y^4 + 0.02786053886x^5y^3 \\
& - 0.03287040141x^4y^4 + 0.02261075355x^5y^2 - 0.01025964786x^4y^3 - 0.03287040141x^3y^4 \\
& - 0.01025964786x^4y^2 + 0.03369734830x^3y^3 - 0.03287040141x^4y - 0.06574080282x^3y^2 \\
& - 0.03287040141x^2y^3 + 0.03369734830x^3y - 0.01025964786x^2y^2 - 0.03287040141x^3 \\
& - 0.01025964786x^2y + 0.02261075355xy^2 - 0.03287040141x^2 + 0.02786053886xy \\
& + 0.02261075355x + 0.02261075355y + 0.02261075355)/(x^3y),
\end{aligned}$$

$$\begin{aligned}
Q_{22}^{(3)}(\omega) = & -(-0.8081660510x^6y^4 - 0.8081660510x^6y^3 - 0.8081660510x^5y^4 + 0.02111418617x^5y^3 \\
& + 0.04628739346x^4y^4 - 0.8081660510x^5y^2 + 0.03820573295x^4y^3 + 0.04628739346x^3y^4 \\
& + 0.03820573295x^4y^2 - 0.1209348371x^3y^3 + 0.04628739346x^4y - 0.9074291765x^3y^2 \\
& + 0.04628739346x^2y^3 - 0.1209348371x^3y + 0.03820573295x^2y^2 + 0.04628739346x^3 \\
& + 0.03820573295x^2y - 0.8081660510xy^2 + 0.04628739346x^2 + 0.02111418617xy \\
& - 0.8081660510x - 0.8081660510y - 0.8081660510)/(x^3y),
\end{aligned}$$

$$\tilde{\mathbf{Q}}^{(1)}(\omega) = \begin{bmatrix} \tilde{\mathbf{Q}}_{11}^{(1)}(\omega) & \tilde{\mathbf{Q}}_{12}^{(1)}(\omega) \\ \tilde{\mathbf{Q}}_{21}^{(1)}(\omega) & \tilde{\mathbf{Q}}_{22}^{(1)}(\omega) \end{bmatrix}$$

$$\tilde{\mathbf{Q}}_{11}^{(1)}(\omega) = (-0.125x^2y^2 + 0.25xy - 0.125)/(x^2y^2),$$

$$\tilde{\mathbf{Q}}_{12}^{(1)}(\omega) = -(0.10279x^2y^2 + 0.10279)/(x^2y^2),$$

$$\tilde{\mathbf{Q}}_{21}^{(1)}(\omega) = 0,$$

$$\tilde{\mathbf{Q}}_{22}^{(1)}(\omega) = (0.0956900x^2y^2 + 0.25xy + 0.0956900)/(x^2y^2),$$

$$\tilde{\mathbf{Q}}^{(2)}(\omega) = \begin{bmatrix} \tilde{\mathbf{Q}}_{11}^{(2)}(\omega) & \tilde{\mathbf{Q}}_{12}^{(2)}(\omega) \\ \tilde{\mathbf{Q}}_{21}^{(2)}(\omega) & \tilde{\mathbf{Q}}_{22}^{(2)}(\omega) \end{bmatrix}$$

$$\tilde{\mathbf{Q}}_{11}^{(2)}(\omega) = -0.125x^2 - 0.125 + 0.25x,$$

$$\tilde{\mathbf{Q}}_{12}^{(2)}(\omega) = -0.125x^2 - 0.125 + 0.25x,$$

$$\tilde{\mathbf{Q}}_{21}^{(2)}(\omega) = 0,$$

$$\tilde{\mathbf{Q}}_{22}^{(2)}(\omega) = 0.095690x^2 + 0.095690 + 0.25x,$$

$$\tilde{\mathbf{Q}}^{(3)}(\omega) = \begin{bmatrix} \tilde{\mathbf{Q}}_{11}^{(3)}(\omega) & \tilde{\mathbf{Q}}_{12}^{(3)}(\omega) \\ \tilde{\mathbf{Q}}_{21}^{(3)}(\omega) & \tilde{\mathbf{Q}}_{22}^{(3)}(\omega) \end{bmatrix}$$

$$\tilde{\mathbf{Q}}_{11}^{(3)}(\omega) = -0.125y^2 - 0.125 + 0.25y,$$

$$\tilde{\mathbf{Q}}_{12}^{(3)}(\omega) = -0.10279y^2 - 0.10279,$$

$$\tilde{\mathbf{Q}}_{21}^{(3)}(\omega) = 0,$$

$$\tilde{\mathbf{Q}}_{22}^{(3)}(\omega) = 0.095690y^2 + 0.095690 + 0.25y,$$

5.1.1 3-Step Multiwavelet Multiresolution Algorithm

For 1×2 row vector $\{\mathbf{v}\}$ (or given triangular mesh C), Fig. 5.9 shows the multiresolution decomposition algorithm, see (5.25)-(5.27), and \mathbf{B} , \mathbf{M} , \mathbf{A} , \mathbf{C} are 2×2 matrices and the entries of these matrices are some constants in \mathbb{R} to be determined.

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

3-step Dyadic Multiwavelet Decomposition Algorithm:

$$\text{Step 1. } \mathbf{v}'' = \{\mathbf{v} - (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) \mathbf{M}\} \mathbf{B}^{-1}; \quad (5.25)$$

$$\text{Step 2. } \tilde{\mathbf{e}} = \mathbf{e} - (\mathbf{v}_0'' + \mathbf{v}_1'') \mathbf{A} - (\mathbf{v}_2'' + \mathbf{v}_3'') \mathbf{C}; \quad (5.26)$$

$$\text{Step 3. } \tilde{\mathbf{v}} = \mathbf{v}'' - (\tilde{\mathbf{e}}_0 + \tilde{\mathbf{e}}_1 + \tilde{\mathbf{e}}_2 + \tilde{\mathbf{e}}_3 + \tilde{\mathbf{e}}_4 + \tilde{\mathbf{e}}_5) \mathbf{W}. \quad (5.27)$$

By the formulas in (5.25), we replace all \mathbf{v} by \mathbf{v}'' . Then, in Step 2, we use the obtained \mathbf{v}'' to find other 3 highpass outputs $\{\tilde{\mathbf{e}}\} (= \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(1)}\} \cup \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(2)}\} \cup \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(3)}\})$ that are associated with edge nodes of \mathcal{M}_0 , by $\tilde{\mathbf{e}}$ given in (5.26). After that in Step 3, we use the obtained $\tilde{\mathbf{e}}$ to update every \mathbf{v}'' in by $\tilde{\mathbf{v}}$ as in (5.27).

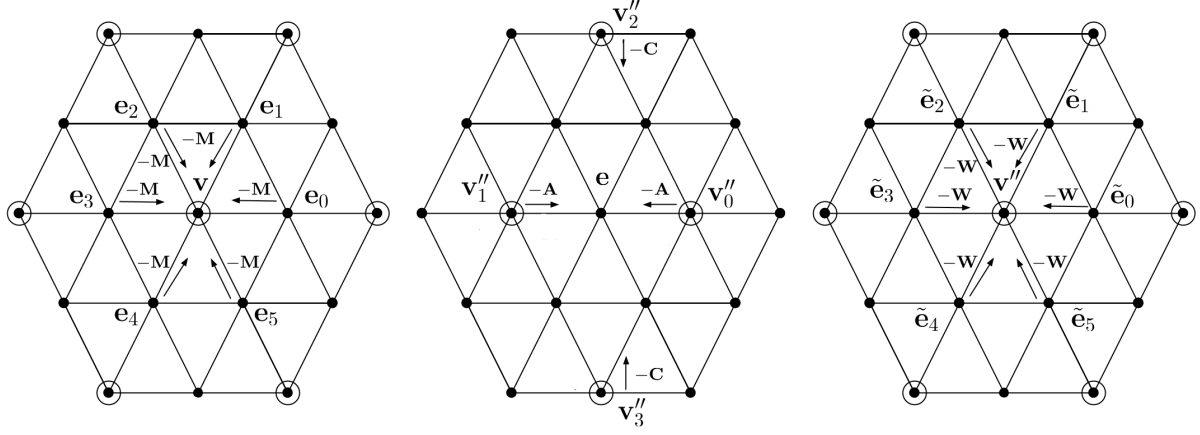


Figure 5.9: Decomposition Algorithm. Left: Step 1; Middle: Step 2; Right: Step 3.

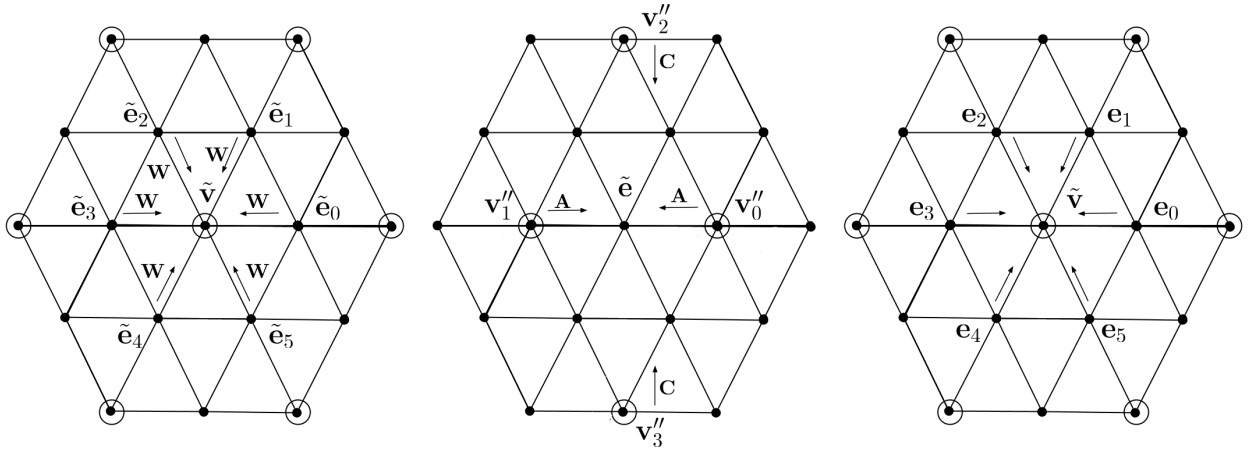


Figure 5.10: Left: Reconstruction Algorithm. Step 1; Middle: Reconstruction Algorithm. Step 2; Right: Reconstruction Step 3.

As in Fig. (5.10), we show the synthesis algorithm and the analysis algorithm. By (5.28)-(5.30), we give the multiresolution reconstruction algorithm, and \mathbf{B} , \mathbf{M} , \mathbf{A} , \mathbf{C} are the same 2×2 matrices we used before.

3-step Dyadic Multiwavelet Reconstruction Algorithm:

$$\text{Step 1. } \mathbf{v}'' = \tilde{\mathbf{v}} + (\tilde{\mathbf{e}}_0 + \tilde{\mathbf{e}}_1 + \tilde{\mathbf{e}}_2 + \tilde{\mathbf{e}}_3 + \tilde{\mathbf{e}}_4 + \tilde{\mathbf{e}}_5)\mathbf{W}; \quad (5.28)$$

$$\text{Step 2. } \mathbf{e} = \tilde{\mathbf{e}} + (\mathbf{v}''_0 + \mathbf{v}''_1)\mathbf{A} + (\mathbf{v}''_2 + \mathbf{v}''_3)\mathbf{C}; \quad (5.29)$$

$$\text{Step 3. } \mathbf{v} = \{\mathbf{v}''\mathbf{B} + (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5)\mathbf{M}\}. \quad (5.30)$$

First in Step 1, we replace the approximation $\tilde{\mathbf{v}}$, which is the detail, by \mathbf{v}'' , respectively given in (5.28). Then, with \mathbf{v}'' , we put \mathbf{e} instead of $\tilde{\mathbf{e}}$, given by (5.29). Next, in Step 3, we use the obtained \mathbf{e} to replace every \mathbf{v}'' by \mathbf{v} as in (5.30). So, the inputs here are $\tilde{\mathbf{v}}$, $\tilde{\mathbf{e}}$ and the outputs are \mathbf{v}'' , \mathbf{v} , \mathbf{e} . As before, \mathbf{B} , \mathbf{M} , \mathbf{A} , \mathbf{C} , \mathbf{W} , \mathbf{U} , are 2×2 matrices and the entries of these matrices are some constants in \mathbb{R} .

We derive the analysis lowpass multifilter \mathbf{P}_{k_1, k_2} which is satisfy the sum rule of order 1, namely:

$$\begin{aligned}
\mathbf{P}_{0,0} &= 4 (\mathbf{B}^{-1})^T + 24 \mathbf{W}^T \mathbf{A}^T (\mathbf{B}^{-1})^T, \\
\mathbf{P}_{1,0} &= \mathbf{P}_{0,1} = \mathbf{P}_{0,-1} = \mathbf{P}_{-1,0} = \mathbf{P}_{-1,-1} = \mathbf{P}_{1,1} = -4 (\mathbf{B}^{-1})^T \mathbf{M}^T - 4 \mathbf{W}^T \\
&\quad - 8 \mathbf{W}^T \mathbf{C}^T (\mathbf{B}^{-1})^T \mathbf{M}^T - 28 \mathbf{W}^T \mathbf{A}^T (\mathbf{B}^{-1})^T \mathbf{M}^T, \\
\mathbf{P}_{2,0} &= \mathbf{P}_{0,2} = \mathbf{P}_{0,-2} = \mathbf{P}_{-2,0} = \mathbf{P}_{-2,-2} = \mathbf{P}_{2,2} = 8 \mathbf{W}^T \mathbf{C}^T (\mathbf{B}^{-1})^T + 4 \mathbf{W}^T \mathbf{A}^T (\mathbf{B}^{-1})^T, \\
\mathbf{P}_{1,2} &= \mathbf{P}_{2,1} = \mathbf{P}_{-1,1} = \mathbf{P}_{1,-1} = \mathbf{P}_{-1,-2} = \mathbf{P}_{-2,-1} = -8 \mathbf{W}^T \mathbf{A}^T (\mathbf{B}^{-1})^T \mathbf{M}^T \\
&\quad - 16 \mathbf{W}^T \mathbf{C}^T (\mathbf{B}^{-1})^T \mathbf{M}^T, \\
\mathbf{P}_{3,0} &= \mathbf{P}_{0,3} = \mathbf{P}_{-3,-3} = \mathbf{P}_{3,3} = \mathbf{P}_{0,-3} = \mathbf{P}_{-3,0} = \mathbf{P}_{1,3} = \mathbf{P}_{3,1} = \mathbf{P}_{2,3} = \mathbf{P}_{3,2} = \mathbf{P}_{-1,-3} \\
&= \mathbf{P}_{-3,-1} = \mathbf{P}_{-2,-3} = \mathbf{P}_{-3,-2} = \mathbf{P}_{-2,1} = \mathbf{P}_{2,-1} = \mathbf{P}_{-1,2} = \mathbf{P}_{1,-2} \\
&= -8 \mathbf{W}^T \mathbf{C}^T (\mathbf{B}^{-1})^T \mathbf{M}^T - 4 \mathbf{W}^T \mathbf{A}^T (\mathbf{B}^{-1})^T \mathbf{M}^T.
\end{aligned}$$

and we derive the synthesis lowpass multifilter $\tilde{\mathbf{P}}_{k_1, k_2}$ which is satisfy the sum rule of order 1, namely:

$$\begin{aligned}
\tilde{\mathbf{P}}_{0,0} &= \mathbf{B} + 6 \mathbf{A} \mathbf{M}, \\
\tilde{\mathbf{P}}_{1,0} &= \tilde{\mathbf{P}}_{0,1} = \tilde{\mathbf{P}}_{0,-1} = \tilde{\mathbf{P}}_{-1,0} = \tilde{\mathbf{P}}_{-1,-1} = \tilde{\mathbf{P}}_{1,1} = \mathbf{A}, \\
\tilde{\mathbf{P}}_{2,0} &= \tilde{\mathbf{P}}_{0,2} = \tilde{\mathbf{P}}_{0,-2} = \tilde{\mathbf{P}}_{-2,0} = \tilde{\mathbf{P}}_{-2,-2} = \tilde{\mathbf{P}}_{2,2} = \mathbf{A} \mathbf{M} + 2 \mathbf{C} \mathbf{M}, \\
\tilde{\mathbf{P}}_{1,2} &= \tilde{\mathbf{P}}_{2,1} = \tilde{\mathbf{P}}_{-1,1} = \tilde{\mathbf{P}}_{1,-1} = \tilde{\mathbf{P}}_{-1,-2} = \tilde{\mathbf{P}}_{-2,-1} = \mathbf{C}.
\end{aligned}$$

We denote

$$\left\{ \begin{array}{l} \mathbf{F}_2(\boldsymbol{\omega}) = \begin{bmatrix} \mathbf{I}_2 & -(1+xy)\mathbf{W}^T & -(1+\frac{1}{x})\mathbf{W}^T & -(1+\frac{1}{y})\mathbf{W}^T \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \\ \mathbf{F}_1(\boldsymbol{\omega}) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -(1+\frac{1}{xy})\mathbf{A}^T - (\frac{1}{x} + \frac{1}{y})\mathbf{C}^T & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -(1+x)\mathbf{A}^T - (xy + \frac{1}{y})\mathbf{C}^T & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ -(1+y)\mathbf{A}^T - (xy + \frac{1}{x})\mathbf{C}^T & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \\ \mathbf{F}_0(\boldsymbol{\omega}) = \begin{bmatrix} (\mathbf{B}^{-1})^T & -(1+xy)(\mathbf{B}^{-1})^T\mathbf{M}^T & -(1+\frac{1}{x})(\mathbf{B}^{-1})^T\mathbf{M}^T & -(1+\frac{1}{y})(\mathbf{B}^{-1})^T\mathbf{M}^T \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}. \end{array} \right. \quad (5.31)$$

and

$$\left\{ \begin{array}{l} \tilde{\mathbf{F}}_2(\boldsymbol{\omega}) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1+\frac{1}{xy})\mathbf{W} & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1+x)\mathbf{W} & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ (1+y)\mathbf{W} & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \\ \tilde{\mathbf{F}}_1(\boldsymbol{\omega}) = \begin{bmatrix} \mathbf{I}_2 & (1+xy)\mathbf{A} + (x+y)\mathbf{C} & (1+\frac{1}{x})\mathbf{A} + (\frac{1}{xy} + y)\mathbf{C} & (1+\frac{1}{y})\mathbf{A} + (\frac{1}{xy} + x)\mathbf{C} \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \\ \tilde{\mathbf{F}}_0(\boldsymbol{\omega}) = \begin{bmatrix} \mathbf{B} & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1+\frac{1}{xy})\mathbf{M} & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1+x)\mathbf{M} & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ (1+y)\mathbf{M} & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}. \end{array} \right. \quad (5.32)$$

The analysis multifilter bank $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}\}$ and the synthesis multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}\}$ can be written as the following:

$$[\mathbf{P}(\boldsymbol{\omega}), \mathbf{Q}^{(1)}(\boldsymbol{\omega}), \mathbf{Q}^{(2)}(\boldsymbol{\omega}), \mathbf{Q}^{(3)}(\boldsymbol{\omega})]^T = \mathbf{F}_2(2\boldsymbol{\omega})\mathbf{F}_1(2\boldsymbol{\omega})\mathbf{F}_0(2\boldsymbol{\omega})\mathbf{I}_0(\boldsymbol{\omega}), \quad (5.33)$$

$$[\tilde{\mathbf{P}}(\omega), \tilde{\mathbf{Q}}^{(1)}(\omega), \tilde{\mathbf{Q}}^{(2)}(\omega), \tilde{\mathbf{Q}}^{(3)}(\omega)]^T = \frac{1}{4} \tilde{\mathbf{F}}_2(2\omega) \tilde{\mathbf{F}}_1(2\omega) \tilde{\mathbf{F}}_0(2\omega) \mathbf{I}_0(\omega), \quad (5.34)$$

where $\mathbf{I}_0(\omega)$ is defined by (5.3).

By solving the system of equations for a sum rule order one both of $\tilde{\mathbf{P}}$ and \mathbf{P} , and for a vanishing moment order one of $\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}$, We have 10 free parameters

$$a_{21}, b_{21}, c_{11}, c_{12}, c_{21}, m_{11}, m_{22}, w_{21}, w_{12}, w_{22}.$$

and we have

$$\begin{aligned} a_{11} &= -c_{11} + \frac{1}{2}, a_{12} = -c_{12}, b_{11} = -6m_{11} + 1, \\ a_{22} &= \frac{1}{96} (48a_{21}c_{11}m_{11}^2 + 48a_{21}c_{11}m_{11}m_{22} - 192a_{21}c_{12}m_{11}w_{21} - 192a_{21}c_{12} \\ &\quad m_{22}w_{21} + 48c_{11}c_{21}m_{11}^2 + 48c_{11}c_{21}m_{11}m_{22} - 192c_{12}c_{21}m_{11}w_{21} - 192c_{12}c_{21} \\ &\quad m_{22}w_{21} + 16a_{21}c_{11}m_{11} + 24a_{21}c_{11}m_{22} + 32a_{21}c_{12}w_{21} + 24a_{21}m_{11}^2 + 48a_{21} \\ &\quad m_{11}m_{22} + 4b_{21}c_{11}m_{11} - 16b_{21}c_{12}w_{21} + 16c_{11}c_{21}m_{11} + 24c_{11}c_{21} \\ &\quad m_{22} + 32c_{12}c_{21}w_{21} + 24c_{21}m_{11}^2 + 24c_{21}m_{11}m_{22} - 4a_{21}c_{11} + 8a_{21}m_{11} + 24a_{21} \\ &\quad m_{22} + 2b_{21}c_{11} + 2b_{21}m_{11} + 4b_{21}m_{22} - 4c_{11}c_{21} + 8c_{21}m_{11} + 12c_{21}m_{22} - 2a_{21} \\ &\quad + b_{21} - 2c_{21}) / (m_{22}w_{21}), \\ b_{12} &= \frac{3}{2} \frac{m_{11}m_{22}}{w_{21}}, m_{12} = -\frac{1}{4} \left(\frac{m_{11}m_{22}}{w_{21}} \right), m_{21} = -4w_{21}, w_{11} = -\frac{1}{4}m_{11} - \frac{1}{8}, \\ b_{22} &= \frac{1}{24} \frac{12a_{21}m_{11} + 12a_{21}m_{22} - 6b_{21}m_{22} + 12c_{21}m_{11} + 12c_{21}m_{22} - 2a_{21} + b_{21} - 2c_{21}}{w_{21}}, \\ c_{22} &= -\frac{1}{96} (48a_{21}c_{11}m_{11}^2 + 48a_{21}c_{11}m_{11}m_{22} - 192a_{21}c_{12}m_{11}w_{21} - 192a_{21}c_{12}m_{22}w_{21} \\ &\quad + 48c_{11}c_{21}m_{11}^2 + 48c_{11}c_{21}m_{11}m_{22} - 192c_{12}c_{21}m_{11}w_{21} - 192c_{12}c_{21}m_{22}w_{21} \\ &\quad + 16a_{21}c_{11}m_{11} + 24a_{21}c_{11}m_{22} + 32a_{21}c_{12}w_{21} + 24a_{21}m_{11}^2 + 24a_{21}m_{11}m_{22} \\ &\quad + 4b_{21}c_{11}m_{11} - 16b_{21}c_{12}w_{21} + 16c_{11}c_{21}m_{11} + 24c_{11}c_{21}m_{22} + 32c_{12}c_{21}w_{21} \\ &\quad + 24c_{21}m_{11}^2 - 4a_{21}c_{11} + 8a_{21}m_{11} + 12a_{21}m_{22} + 2b_{21}c_{11} + 2b_{21}m_{11} + 2b_{21}m_{22} \\ &\quad - 4c_{11}c_{21} + 8c_{21}m_{11} - 2a_{21} + b_{21} - 2c_{21}) / (m_{22}w_{21}). \end{aligned}$$

If we choose

$$\begin{aligned} a_{21} &= -\frac{316}{819}, b_{21} = \frac{13}{184}, c_{11} = \frac{31}{246}, c_{12} = \frac{1}{4699}, c_{21} = \frac{749}{1111}, m_{11} = \frac{2}{19}, m_{22} = -\frac{20}{813}, \\ w_{21} &= \frac{581}{269}, w_{12} = \frac{291}{427}, w_{22} = \frac{47}{162}. \end{aligned}$$

By using the smoothness formula, we can obtain $\tilde{\Phi} \in W^2$, but $\Phi \notin L^2(\mathbb{R}^2)^2$.

If we choose another parameters

$$\begin{aligned} a_{21} &= -\frac{142}{29}, b_{21} = \frac{3088}{361}, c_{11} = \frac{-20}{251}, c_{12} = \frac{-1}{499}, c_{21} = \frac{-54}{247}, m_{11} = \frac{-272}{653}, m_{22} = -\frac{100}{81}, \\ w_{21} &= \frac{16313}{2039}, w_{12} = \frac{-4}{177}, w_{22} = \frac{-162}{197}. \end{aligned}$$

By using the smoothness formula, we can obtain $\tilde{\Phi} \in W^{0.8995}$ and $\Phi \in W^{0.0107}$. We can choose another values of the parameters such that Φ is in $L^2(\mathbb{R}^2)^2$, but we cannot get smoother $\tilde{\Phi}$ as long as its dual is in $L^2(\mathbb{R}^2)^2$. The lowpass analysis mask \mathbf{P} and the lowpass synthesis mask $\tilde{\mathbf{P}}$ will be as follows:

$$\mathbf{P}(\omega) = \begin{bmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} P_{11}(\omega) = & (0.02480353830x^5y^5 + 0.1250050698x^4y^4 - 0.04960608426x^3y^5 \\ & + 0.1250050698x^4y^3 + 0.1250050698x^3y^4 + 0.3988214460x^3y^3 + 0.1250050698x^3y^2 \\ & + 0.1250050698x^2y^3 - 0.02480353830x^3y + 0.1250050698x^2y^2 - 0.02480353830xy^3 \\ & - 0.02480353830xy)/(x^3y^3), \end{aligned}$$

$$\begin{aligned} P_{12}(\omega) = & (0.6429370860x^5y^5 + 1.285874172x^3y^5 - 3.857622516x^3y^3 + 0.6429370860x^3y \\ & + 0.6429370860xy^3 + 0.6429370860xy)/(x^3y^3), \end{aligned}$$

$$\begin{aligned} P_{21}(\omega) = & (0.0002340433890x - 0.0002340433890y - 0.0002340433890 - 0.0004680768546x^4y^2 \\ & - 0.0004680768546x^2y^4 - 0.0002340433890x^4y - 0.0002340433890xy^4 \\ & - 0.0004680768546x^2y - 0.0004680768546xy^2 - 0.0002340433890x^6y^6 \\ & - 0.0002340433890x^6y^5 - 0.0002340433890x^5y^6 - 0.0002340433890x^6y^4 \\ & - 0.0002340433890x^4y^6 - 0.0002340433890x^6y^3 - 0.0047169x^5y^4 - 0.0047169x^4y^5 \\ & - 0.0002340433890x^3y^6 - 0.0002340433890x^5y^2 - 0.0023585x^2y^5 - 0.0002340433890x^2 \\ & - 0.0002340433890y^2 - 0.0002340433890x^3 - 0.0002340433890y^3 - 0.006446538342x^5y^5 \\ & + 0.02440660230x^4y^4 - 0.01289347362x^3y^5 + 0.02440660230x^4y^3 + 0.02440660230x^3y^4 \\ & - 0.1007423568x^3y^3 + 0.02440660230x^3y^2 + 0.02440660230x^2y^3 - 0.006446538342x^3y \\ & + 0.02440660230x^2y^2 - 0.006446538342xy^3 - 0.006446538342xy)/(x^3y^3), \end{aligned}$$

$$\begin{aligned}
P_{22}(\omega) = & (0.00685850x - 0.01798120080y - 0.01798120080 - 0.03596140926x^4y^2 \\
& - 0.03596140926x^2y^4 - 0.01798120080x^4y - 0.01798120080xy^4 - 0.03596140926x^2y \\
& - 0.03596140926xy^2 - 0.01798120080x^6y^6 - 0.01798120080x^6y^5 - 0.01798120080x^5y^6 \\
& - 0.01798120080x^6y^4 - 0.01798120080x^4y^6 - 0.01798120080x^6y^3 - 0.03596140926x^5y^4 \\
& - 0.03596140926x^4y^5 - 0.01798120080x^3y^6 - 0.01798120080x^5y^2 - 0.01798120080x^2y^5 \\
& - 0.01798120080x^2 - 0.01798120080y^2 - 0.01798120080x^3 - 0.01798120080y^3 \\
& + 0.1525425048x^5y^5 + 0.9612301410x^4y^4 + 3.0744x^3y^5 + 0.9612301410x^4y^3 \\
& + 0.9612301410x^3y^4 + 27.595x^3y^3 + 0.9612301410x^3y^2 + 0.9612301410x^2y^3 \\
& + 0.1525425048x^3y + 0.9612301410x^2y^2 + 0.1525425048xy^3 + 0.1525425048xy)/(x^3y^3).
\end{aligned}$$

and

$$\tilde{\mathbf{P}}(\omega) = \begin{bmatrix} \tilde{P}_{11}(\omega) & \tilde{P}_{12}(\omega) \\ \tilde{P}_{21}(\omega) & \tilde{P}_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned}
\tilde{P}_{11}(\omega) = & (0.0277375x^4y^4 - 0.01992025x^4y^3 - 0.01992025x^3y^4 - 0.0277375x^4y^2 \\
& + 0.1449200x^3y^3 - 0.0277375x^2y^4 + 0.1449200x^3y^2 + 0.1449200x^2y^3 \\
& - 0.01992025x^3y + 0.416425x^2y^2 - 0.01992025xy^3 + 0.1449200x^2y \\
& + 0.1449200xy^2 - 0.0277375x^2 + 0.1449200xy - 0.0277375y^2 \\
& - 0.01992025x - 0.01992025y - 0.0277375)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_{12}(\omega) = & (0.001070025x^4y^4 - 0.000501000x^4y^3 - 0.000501000x^3y^4 - 0.001070025x^4y^2 \\
& + 0.000501000x^3y^3 - 0.001070025x^2y^4 + 0.000501000x^3y^2 + 0.000501000x^2y^3 \\
& - 0.000501000x^3y + 0.00642025x^2y^2 - 0.000501000xy^3 + 0.000501000x^2y \\
& + 0.000501000xy^2 - 0.001070025x^2 + 0.000501000xy - 0.001070025y^2 \\
& - 0.000501000x - 0.000501000y - 0.001070025)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_{21}(\omega) = & (0.264475x^4y^4 - 0.0546550x^4y^3 - 0.0546550x^3y^4 \\
& + 0.264475x^4y^2 - 1.224150x^3y^3 + 0.264475x^2y^4 - 1.224150x^3y^2 \\
& - 1.224150x^2y^3 - 0.0546550x^3y + 6.08600x^2y^2 - 0.0546550xy^3 \\
& - 1.224150x^2y - 1.224150xy^2 + 0.264475x^2 - 1.224150xy + 0.264475y^2 \\
& - 0.0546550x - 0.0546550y + 0.264475)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_{22}(\omega) = & (0.01020275x^4y^4 + 0.00685850x^4y^3 + 0.00685850x^3y^4 + 0.01020275x^4y^2 \\
& - 0.00462500x^3y^3 + 0.01020275x^2y^4 - 0.00462500x^3y^2 - 0.00462500x^2y^3 \\
& + 0.00685850x^3y + 0.391200x^2y^2 + 0.027434xy^3 - 0.00685850x^2y \\
& - 0.00462500xy^2 + 0.01020275x^2 - 0.00462500xy + 0.01020275y^2 \\
& + 0.00685850x + 0.00685850y + 0.01020275)/(x^2y^2).
\end{aligned}$$

The highpass masks are

$$\begin{aligned}
Q_{11}^{(1)}(\omega) = & -(0.02873329578x^4y^4 + 0.02873329578x^4y^3 + 0.02873329578x^3y^4 \\
& - 0.00963730626x^4y^2 + 0.40730013x^3y^3 - 0.00963730626x^2y^4 - 0.00963730626x^4y \\
& + 0.01909598952x^3y^2 + 0.01909598952x^2y^3 - 0.00963730626xy^4 - 0.02873329578x^3y \\
& - 0.942534030x^2y^2 - 0.02873329578xy^3 - 0.00963730626x^3 + 0.01909598952x^2y \\
& + 0.01909598952xy^2 - 0.00963730626y^3 - 0.00963730626x^2 + 0.40730013xy \\
& - 0.00963730626y^2 + 0.02873329578x + 0.02873329578y + 0.02873329578)/(x^3y^3),
\end{aligned}$$

$$\begin{aligned}
Q_{12}^{(1)}(\omega) = & -(2.207502422x^4y^4 + 2.207502422x^4y^3 + 2.207502422x^3y^4 \\
& - 0.740412182x^4y^2 - 8.76959542x^3y^3 - 0.740412182x^2y^4 - 0.740412182x^4y \\
& + 1.467090240x^3y^2 + 1.467090240x^2y^3 - 0.740412182xy^4 - 0.03311273480x^3y \\
& + 1.467090240x^2y^2 - 0.03311273480xy^3 - 0.740412182x^3 + 1.467090240x^2y \\
& + 1.467090240xy^2 - 0.740412182y^3 - 0.740412182x^2 - 8.76959542xy \\
& - 0.740412182y^2 + 2.207502422x + 2.207502422y + 2.207502422)/(x^3y^3),
\end{aligned}$$

$$\begin{aligned}
Q_{21}^{(1)}(\omega) = & -(0.0000893075306x^4y^4 + 0.0000893075306x^4y^3 + 0.0000893075306x^3y^4 \\
& - 0.00003231964868x^4y^2 + 0.001468201672x^3y^3 - 0.00003231964868x^2y^4 \\
& - 0.00003231964868x^4y + 0.0000569886758x^3y^2 + 0.0000569886758x^2y^3 \\
& - 0.00003231964868xy^4 - 0.001810125788x^3y + 0.0001786150612x^2y^2 \\
& - 0.001810125788xy^3 - 0.00003231964868x^3 + 0.0000569886758x^2y \\
& + 0.0000569886758xy^2 - 0.00003231964868y^3 - 0.00003231964868x^2 \\
& + 0.001468201672xy - 0.00003231964868y^2 + 0.0000893075306x \\
& + 0.0000893075306y + 0.0000893075306)/(x^3y^3),
\end{aligned}$$

$$\begin{aligned}
Q_{22}^{(1)}(\omega) = & -(0.00686150484x^4y^4 + 0.00686150484x^4y^3 + 0.00686150484x^3y^4 \\
& - 0.002483058170x^4y^2 - 0.03249985944x^3y^3 - 0.002483058170x^2y^4 - 0.002483058170x^4y \\
& + 0.00437824820x^3y^2 + 0.00437824820x^2y^3 - 0.002483058170xy^4 + 0.0449097916x^3y \\
& - 0.986276818x^2y^2 + 0.0449097916xy^3 - 0.002483058170x^3 + 0.00437824820x^2y \\
& + 0.00437824820xy^2 - 0.002483058170y^3 - 0.002483058170x^2 - 0.03249985944xy \\
& - 0.002483058170y^2 + 0.00686150484x + 0.00686150484y + 0.00686150484)/(x^3y^3).
\end{aligned}$$

$$\mathbf{Q}^{(2)}(\omega) = \begin{bmatrix} Q_{11}^{(2)}(\omega) & Q_{12}^{(2)}(\omega) \\ Q_{21}^{(2)}(\omega) & Q_{22}^{(2)}(\omega) \end{bmatrix}$$

where

$$\begin{aligned}
Q_{11}^{(2)}(\omega) = & -(-0.00963730626x^4y^6 - 0.00963730626x^4y^5 - 0.00963730626x^3y^6 \\
& + 0.02873329578x^4y^4 - 0.02185908886x^3y^5 + 0.02873329578x^4y^3 + 0.01909598952x^3y^4 \\
& - 0.00963730626x^2y^5 + 0.407300134x^3y^3 + 0.01909598952x^2y^4 + 0.02873329578x^3y^2 \\
& - 0.942534030x^2y^3 + 0.02873329578xy^4 + 0.01909598952x^2y^2 + 0.407300134xy^3 \\
& - 0.00963730626x^2y + 0.01909598952xy^2 + 0.02873329578y^3 - 0.02185908886xy \\
& + 0.02873329578y^2 - 0.00963730626x - 0.00963730626y - 0.00963730626)/(xy^3),
\end{aligned}$$

$$\begin{aligned}
Q_{12}^{(2)}(\omega) = & -(-0.740412182x^4y^6 - 0.740412182x^4y^5 - 0.740412182x^3y^6 \\
& + 2.207502422x^4y^4 - 0.03311273480x^3y^5 + 2.207502422x^4y^3 + 1.467090240x^3y^4 \\
& - 0.740412182x^2y^5 - 8.76959542x^3y^3 + 1.467090240x^2y^4 + 2.207502422x^3y^2 \\
& + 4.41516362x^2y^3 + 2.207502422xy^4 + 1.467090240x^2y^2 - 8.76959542xy^3 \\
& - 0.740412182x^2y + 1.467090240xy^2 + 2.207502422y^3 - 0.03311273480xy \\
& + 2.207502422y^2 - 0.740412182x - 0.740412182y - 0.740412182)/(xy^3),
\end{aligned}$$

$$\begin{aligned}
Q_{21}^{(2)}(\omega) = & -(-0.00003231964868x^4y^6 - 0.00003231964868x^4y^5 - 0.00003231964868x^3y^6 \\
& + 0.0000893075306x^4y^4 - 0.001810125788x^3y^5 + 0.0000893075306x^4y^3 \\
& + 0.0000893075306x^3y^4 - 0.00003231964868x^2y^5 + 0.00003231964868x^3y^3 \\
& + 0.0000893075306x^2y^4 + 0.0000893075306x^3y^2 + 0.00044998x^2y^3 \\
& + 0.0000893075306xy^4 + 0.0000893075306x^2y^2 + 0.00003231964868xy^3 \\
& - 0.00003231964868x^2y + 0.0000893075306xy^2 + 0.0000893075306y^3 \\
& - 0.001810125788xy + 0.0000893075306y^2 - 0.00003231964868x \\
& - 0.00003231964868y - 0.00003231964868)/(xy^3),
\end{aligned}$$

$$\begin{aligned}
Q_{22}^{(2)}(\omega) = & -(-0.002483058170x^4y^6 - 0.002483058170x^4y^5 - 0.002483058170x^3y^6 \\
& + 0.00686150484x^4y^4 + 0.00686150484x^3y^5 + 0.00686150484x^4y^3 + 0.011030x^3y^4 \\
& - 0.002483058170x^2y^5 - 0.03249985944x^3y^3 + 0.011030x^2y^4 + 0.00686150484x^3y^2 \\
& - 0.986276818x^2y^3 + 0.00686150484xy^4 + 0.011030x^2y^2 - 0.03249985944xy^3 \\
& - 0.002483058170x^2y + 0.011030xy^2 + 0.00686150484y^3 + 0.00686150484xy \\
& + 0.00686150484y^2 - 0.002483058170x - 0.002483058170y - 0.002483058170)/(xy^3).
\end{aligned}$$

$$\mathbf{Q}^{(3)}(\omega) = \begin{bmatrix} Q_{11}^{(3)}(\omega) & Q_{12}^{(3)}(\omega) \\ Q_{21}^{(3)}(\omega) & Q_{22}^{(3)}(\omega) \end{bmatrix}$$

where

$$\begin{aligned}
Q_{11}^{(3)}(\omega) = & -(-0.00963730626x^6y^4 + 1.000010942x^5y^3 - 0.00963730626x^6y^3 \\
& -0.00963730626x^5y^4 + 0.02873329578x^4y^4 - 0.00963730626x^5y^2 + 0.01909598952x^4y^3 \\
& +0.02873329578x^3y^4 + 0.01909598952x^4y^2 + 0.407300134x^3y^3 + 0.02873329578x^4y \\
& -0.942534030x^3y^2 + 0.02873329578x^2y^3 + 0.407300134x^3y + 0.01909598952x^2y^2 \\
& +0.02873329578x^3 + 0.01909598952x^2y - 0.00963730626xy^2 + 0.02873329578x^2 \\
& -0.02185908886xy - 0.00963730626x - 0.00963730626y - 0.00963730626)/(x^3y),
\end{aligned}$$

$$\begin{aligned}
Q_{12}^{(3)}(\omega) = & -(-0.740412182x^6y^4 - 0.740412182x^6y^3 - 0.740412182x^5y^4 \\
& +2.207502422x^4y^4 - 0.740412182x^5y^2 + 0.740412182x^4y^3 + 2.207502422x^3y^4 \\
& +0.740412182x^4y^2 - 8.76959542x^3y^3 + 2.207502422x^4y + 4.41516362x^3y^2 \\
& +2.207502422x^2y^3 - 8.76959542x^3y + 0.740412182x^2y^2 + 2.207502422x^3 \\
& +0.740412182x^2y - 0.740412182xy^2 + 2.207502422x^2 - 0.03311273480xy \\
& -0.740412182x - 0.740412182y - 0.740412182)/(x^3y),
\end{aligned}$$

$$\begin{aligned}
Q_{21}^{(3)}(\omega) = & -(-0.00003231964868x^6y^4 - 0.00003231964868x^6y^3 - 0.00003231964868x^5y^4 \\
& +0.0000893075306x^4y^4 - 0.00003231964868x^5y^2 + 0.0000569886758x^4y^3 \\
& +0.0000893075306x^3y^4 + 0.0000569886758x^4y^2 + 0.00003231964868x^3y^3 \\
& +0.0000893075306x^4y + 0.00044998x^3y^2 + 0.0000893075306x^2y^3 \\
& +0.00003231964868x^3y + 0.0000569886758x^2y^2 + 0.0000893075306x^3 \\
& +0.0000569886758x^2y - 0.00003231964868xy^2 + 0.0000893075306x^2 \\
& -0.001810125788xy - 0.00003231964868x - 0.00003231964868y \\
& -0.00003231964868)/(x^3y),
\end{aligned}$$

$$\begin{aligned}
Q_{22}^{(3)}(\omega) = & -(-0.002483058170x^6y^4 + 1.000010942x^5y^3 - 0.002483058170x^6y^3 \\
& -0.002483058170x^5y^4 + 0.00686150484x^4y^4 - 0.002483058170x^5y^2 + 0.00686150484x^4y^3 \\
& +0.00686150484x^3y^4 + 0.00686150484x^4y^2 - 0.03249985944x^3y^3 + 0.00686150484x^4y \\
& -2.4847x^3y^2 + 0.00686150484x^2y^3 - 0.03249985944x^3y + 0.00686150484x^2y^2 \\
& +0.00686150484x^3 + 0.00686150484x^2y - 0.002483058170xy^2 + 0.00686150484x^2 \\
& +0.0449097916xy - 0.002483058170x - 0.002483058170y - 0.002483058170)/(x^3y),
\end{aligned}$$

$$\tilde{\mathbf{Q}}^{(1)}(\omega) = \begin{bmatrix} \tilde{\mathbf{Q}}_{11}^{(1)}(\omega) & \tilde{\mathbf{Q}}_{12}^{(1)}(\omega) \\ \tilde{\mathbf{Q}}_{21}^{(1)}(\omega) & \tilde{\mathbf{Q}}_{22}^{(1)}(\omega) \end{bmatrix}$$

$$\begin{aligned}\tilde{Q}_{11}^{(1)}(\omega) = & (0.001650800x^5y^3 + 0.001650800x + 0.001650800y - 0.00539800 - 0.01079600x^4y^2 \\ & - 0.01079600x^2y^4 + 0.02464025x^2y + 0.02464025xy^2 - 0.00539800x^6y^6 + 0.001650800x^6y^5 \\ & + 0.001650800x^5y^6 - 0.00539800x^6y^4 - 0.00539800x^4y^6 + 0.02464025x^5y^4 + 0.02464025x^4y^5 \\ & - 0.00997225x^2 - 0.00997225y^2 + 0.02464025x^5y^5 - 0.255750x^4y^4 + 0.001650800x^3y^5 \\ & + 0.0262900x^4y^3 + 0.0262900x^3y^4 + 1.1971x^3y^3 + 0.0262900x^3y^2 \\ & + 0.0262900x^2y^3 + 0.001650800x^3y - 0.255750x^2y^2 + 0.006603xy^3 \\ & + 0.02464025xy)/(x^4y^4),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{12}^{(1)}(\omega) = & (-0.001650800x^5y^3 - 0.001650800x - 0.001650800y - 0.00539800 \\ & - 0.01079600x^4y^2 - 0.01079600x^2y^4 + 0.02464025x^2y + 0.02464025xy^2 \\ & - 0.00539800x^6y^6 - 0.001650800x^6y^5 - 0.001650800x^5y^6 - 0.00539800x^6y^4 \\ & - 0.00539800x^4y^6 + 0.02464025x^5y^4 + 0.02464025x^4y^5 - 0.0032046x^2 \\ & - 0.0032046y^2 + 0.02464025x^5y^5 - 0.255750x^4y^4 - 0.001650800x^3y^5 \\ & - 0.0262900x^4y^3 - 0.0262900x^3y^4 + 0.299275x^3y^3 - 0.0262900x^3y^2 \\ & - 0.0262900x^2y^3 - 0.001650800x^3y - 0.255750x^2y^2 - 0.001650800xy^3 \\ & + 0.02464025xy)/(x^4y^4),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{21}^{(1)}(\omega) = & (-0.1144275x^5y^3 - 0.1144275x - 0.1144275y - 0.878775x^4y^2 - 0.878775x^2y^4 \\ & + 2.166075x^2y + 2.166075xy^2 - 0.439400x^6y^6 - 0.1144275x^6y^5 - 0.1144275x^5y^6 \\ & - 0.439400x^6y^4 - 0.439400x^4y^6 + 2.166075x^5y^4 + 2.166075x^4y^5 - 1.381100x^2 \\ & - 1.381100y^2 + 2.166075x^5y^5 - 10.11300x^4y^4 - 0.1144275x^3y^5 + 2.051650x^4y^3 \\ & + 2.051650x^3y^4 + 4.33225x^3y^3 + 2.051650x^3y^2 + 2.051650x^2y^3 \\ & - 0.1144275x^3y - 10.11300x^2y^2 - 0.1144275xy^3 + 2.166075xy - 0.439400)/(x^4y^4),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{22}^{(1)}(\omega) = & (-0.0024120625x^5y^3 - 0.0024120625x - 0.0024120625y - 0.0042376875 \\ & - 0.0339025x^4y^2 - 0.0339025x^2y^4 + 0.0019528750x^2y + 0.0019528750xy^2 \\ & - 0.0042376875x^6y^6 - 0.0024120625x^6y^5 - 0.0024120625x^5y^6 - 0.0042376875x^6y^4 \\ & - 0.0042376875x^4y^6 + 0.0019528750x^5y^4 + 0.0019528750x^4y^5 - 0.0637475x^2 \\ & - 0.0637475y^2 + 0.0019528750x^5y^5 - 0.5959257x^4y^4 - 0.0024120625x^3y^5 \\ & - 0.001836750x^4y^3 - 0.001836750x^3y^4 + 0.066406375x^3y^3 \\ & - 0.001836750x^3y^2 - 0.001836750x^2y^3 - 0.0024120625x^3y \\ & - 0.5959257x^2y^2 - 0.0024120625xy^3 + 0.0019528750xy)/(x^4y^4),\end{aligned}$$

$$\tilde{\mathbf{Q}}^{(2)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(2)}(\omega) & \tilde{Q}_{12}^{(2)}(\omega) \\ \tilde{Q}_{21}^{(2)}(\omega) & \tilde{Q}_{22}^{(2)}(\omega) \end{bmatrix}$$

$$\begin{aligned}
\tilde{Q}_{11}^{(2)}(\omega) = & (0.001650800x^3 + 0.299275x^3y^2 - 0.255750x^2y^2 - 0.00539800x^6y^4 + 0.001650800x^6y^3 \\
& + 0.001650800x^5y^4 - 0.00539800x^6y^2 + 0.02464025x^5y^3 - 0.01079600x^4y^4 \\
& + 0.02464025x^5y^2 + 0.0262900x^4y^3 + 0.001650800x^3y^4 + 0.001650800x^5y \\
& - 0.255750x^4y^2 + 0.0262900x^3y^3 - 0.00539800x^2y^4 + 0.02464025x^4y \\
& + 0.02464025x^2y^3 + 0.0262900x^3y + 0.001650800xy^3 + 0.0262900x^2y \\
& + 0.02464025xy^2 + 0.02464025xy - 0.01079600x^2 - 0.00539800x^4 \\
& - 0.00539800y^2 + 0.001650800y + 0.001650800x - 0.00539800)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{12}^{(2)}(\omega) = & (-0.0001445400x^3 + 0.0001881275x^3y^2 - 0.01320000x^2y^2 - 0.00083297x^6y^4 \\
& - 0.0001445400x^6y^3 - 0.0001445400x^5y^4 - 0.00083297x^6y^2 + 0.0000940625x^5y^3 \\
& - 0.000416475x^4y^4 + 0.0000940625x^5y^2 - 0.0000504775x^4y^3 - 0.0001445400x^3y^4 \\
& - 0.0001445400x^5y - 0.01320000x^4y^2 - 0.0000504775x^3y^3 - 0.00083297x^2y^4 \\
& + 0.0000940625x^4y + 0.0000940625x^2y^3 - 0.0000504775x^3y - 0.0001445400xy^3 \\
& - 0.0000504775x^2y + 0.0000940625xy^2 + 0.0000940625xy - 0.000416475x^2 \\
& - 0.00083297x^4 - 0.00083297y^2 - 0.0001445400y - 0.0001445400x \\
& - 0.00083297)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{21}^{(2)}(\omega) = & (-0.1144275x^3 + 4.33225x^3y^2 - 10.11300x^2y^2 - 0.439400x^6y^4 \\
& - 0.1144275x^6y^3 - 0.1144275x^5y^4 - 0.439400x^6y^2 + 2.166075x^5y^3 \\
& - 0.878775x^4y^4 + 2.166075x^5y^2 + 2.051650x^4y^3 - 0.1144275x^3y^4 \\
& - 0.1144275x^5y - 10.11300x^4y^2 + 2.051650x^3y^3 - 0.439400x^2y^4 \\
& + 2.166075x^4y + 2.166075x^2y^3 + 2.051650x^3y - 0.1144275xy^3 \\
& + 2.051650x^2y + 2.166075xy^2 + 2.166075xy - 0.878775x^2 - 0.439400 \\
& - 0.439400x^4 - 0.439400y^2 - 0.1144275y - 0.1144275x)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{22}^{(2)}(\omega) = & (-0.00964825x^3 + 0.265625x^3y^2 - 0.595925x^2y^2 - 0.0042376875x^6y^4 \\
& - 0.00964825x^6y^3 - 0.00964825x^5y^4 - 0.0042376875x^6y^2 + 0.00781150x^5y^3 \\
& - 0.0339025x^4y^4 + 0.00781150x^5y^2 - 0.00045918750x^4y^3 - 0.00964825x^3y^4 \\
& - 0.00964825x^5y - 0.595925x^4y^2 - 0.00045918750x^3y^3 - 0.0042376875x^2y^4 \\
& + 0.00781150x^4y + 0.00781150x^2y^3 - 0.00045918750x^3y - 0.00964825xy^3 \\
& - 0.00045918750x^2y + 0.00781150xy^2 + 0.00781150xy - 0.0339025x^2 \\
& - 0.0042376875x^4 - 0.0042376875y^2 - 0.00964825y - 0.00964825x \\
& - 0.0042376875)/(x^2y^2),
\end{aligned}$$

$$\tilde{\mathbf{Q}}^{(3)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(3)}(\omega) & \tilde{Q}_{12}^{(3)}(\omega) \\ \tilde{Q}_{21}^{(3)}(\omega) & \tilde{Q}_{22}^{(3)}(\omega) \end{bmatrix}$$

$$\begin{aligned} \tilde{Q}_{11}^{(3)}(\omega) = & (0.01079600y^3 + 0.02464025x^3y^2 - 0.255750x^2y^2 - 0.01079600x^4y^4 \\ & + 0.01079600x^4y^3 + 0.299275x^3y^4 - 0.021592x^4y^2 + 0.299275x^3y^3 - 0.255750x^2y^4 \\ & + 1.1971x^2y^3 + 0.01079600x^3y + 0.299275xy^3 + 0.02464025x^2y + 0.299275xy^2 \\ & + 0.02464025xy - 0.021592x^4y^6 + 0.01079600x^4y^5 + 0.01079600x^3y^6 \\ & + 0.02464025x^3y^5 - 0.021592x^2y^6 + 0.02464025x^2y^5 + 0.01079600xy^5 \\ & + 0.02464025xy^4 - 0.021592x^2 - 0.01079600y^2 - 0.021592y^4 \\ & + 0.01079600y + 0.01079600x - 0.021592)/(x^2y^2), \end{aligned}$$

$$\begin{aligned} \tilde{Q}_{12}^{(3)}(\omega) = & (-0.00057816y^3 + 0.00037625x^3y^2 - 0.052800x^2y^2 - 0.0016659x^4y^4 \\ & - 0.00057816x^4y^3 - 0.00020191x^3y^4 - 0.00083297x^4y^2 - 0.00020191x^3y^3 \\ & - 0.052800x^2y^4 + 0.00075251x^2y^3 - 0.00057816x^3y - 0.00020191xy^3 \\ & + 0.00037625x^2y - 0.00020191xy^2 + 0.00037625xy - 0.00083297x^4y^6 \\ & - 0.00057816x^4y^5 - 0.00057816x^3y^6 + 0.00037625x^3y^5 - 0.00083297x^2y^6 \\ & + 0.00037625x^2y^5 - 0.00057816xy^5 + 0.00037625xy^4 - 0.00083297x^2 \\ & - 0.0016659y^2 - 0.00083297y^4 - 0.00057816y - 0.00057816x \\ & - 0.00083297)/(x^2y^2), \end{aligned}$$

$$\begin{aligned} \tilde{Q}_{21}^{(3)}(\omega) = & (-0.1144275y^3 + 2.166075x^3y^2 - 10.11300x^2y^2 - 0.878775x^4y^4 - 0.1144275x^4y^3 \\ & + 8.2066x^3y^4 - 0.255750x^4y^2 + 8.2066x^3y^3 - 10.11300x^2y^4 + 0.299275x^2y^3 \\ & - 0.1144275x^3y + 8.2066xy^3 + 2.166075x^2y + 8.2066xy^2 + 2.166075xy \\ & - 0.255750x^4y^6 - 0.1144275x^4y^5 - 0.1144275x^3y^6 + 2.166075x^3y^5 \\ & - 0.255750x^2y^6 + 2.166075x^2y^5 - 0.1144275xy^5 + 2.166075xy^4 \\ & - 0.255750x^2 - 0.255750 - 0.878775y^2 - 0.255750y^4 - 0.1144275y \\ & - 0.1144275x)/(x^2y^2), \end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{22}^{(3)}(\omega) = & (-0.00964825y^3 + 0.00781150x^3y^2 - 0.595925x^2y^2 - 0.0339025x^4y^4 \\
& -0.00964825x^4y^3 - 0.00045918750x^3y^4 - 0.595925x^4y^2 - 0.00045918750x^3y^3 \\
& -0.595925x^2y^4 + 0.265625x^2y^3 - 0.00964825x^3y - 0.00045918750xy^3 \\
& +0.00781150x^2y - 0.00045918750xy^2 + 0.00781150xy - 0.595925x^4y^6 \\
& -0.00964825x^4y^5 - 0.00964825x^3y^6 + 0.00781150x^3y^5 - 0.595925x^2y^6 \\
& +0.00781150x^2y^5 - 0.00964825xy^5 + 0.00781150xy^4 - 0.595925x^2 \\
& -0.595925 - 0.013561y^2 - 0.595925y^4 - 0.00964825y - 0.00964825x)/(x^2y^2),
\end{aligned}$$

Chapter 6

Two-dimensional Multiple Bi-frames and Multiresolution Algorithms

Two-dimensional wavelets play an important role in several applications, including image compression. In this chapter, we will extend the work in [76] from (the scalar case) to (the vector case), which studies the biorthogonal wavelet frames for triangular mesh-based surface multiresolution (multiscale) processing. We will show how to represent multiple bi-frame multiresolution algorithms for regular vertices as templates. Then, they are implementable. This will give an important way to the approach of template-based multiple bi-frame construction.

In Section 6.1, we introduce some important notations and definitions in multiwavelet case that will be used later in this section. In Section 6.2, we develop 6-fold symmetry dyadic multiple bi-frames and associated templates. Also, we present and obtain some results of a 2-step multiple bi-frame multiresolution algorithm. Then, we develop and obtain some results of a 3-step multiple bi-frame multiresolution algorithm. Finally, we use our multiple bi-frame algorithm for surface noise-removing.

6.1 Introduction

We consider a two-dimensional (2-D for short) multiple bi-frame multiresolution algorithm and we study multiple bi-frames with 4 frame generators. In this section, the multiplicity $r = 2$. So, the multiscaling function and the multiwavelets are 2×1 column vector functions and the matrix masks are 2×2 square matrices. We use \mathbf{k} to denote elements of \mathbf{Z}^2 and $\boldsymbol{\omega}$, \mathbf{x} to denote elements of \mathbb{R}^2 . The multi-indices $\mathbf{k} \in \mathbf{Z}^2$ and $\boldsymbol{\omega}$, $\mathbf{x} \in \mathbb{R}^2$ are written as row vectors

$$\mathbf{k} = (k_1, k_2), \mathbf{x} = (x_1, x_2), \boldsymbol{\omega} = (\omega_1, \omega_2).$$

Let $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ denote the associated refinable vector functions satisfying the refinement equations

$$\boldsymbol{\Phi}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbf{Z}^2} \mathbf{P}_{\mathbf{k}} \boldsymbol{\Phi}(2\mathbf{x} - \mathbf{k}), \quad \tilde{\boldsymbol{\Phi}}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbf{Z}^2} \tilde{\mathbf{P}}_{\mathbf{k}} \tilde{\boldsymbol{\Phi}}(2\mathbf{x} - \mathbf{k}), \quad \mathbf{x} \in \mathbb{R}^2. \quad (6.1)$$

The multi-indices $\mathbf{k} = (k_1, k_2)$ are the indices for the coefficients $\mathbf{P}_{\mathbf{k}}$.

For a matrix sequence $\{\mathbf{P}_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^2}$ which is called a matrix refinement sequence of 2×2 matrices with finite nonzero $\mathbf{P}_{\mathbf{k}}$. Let $\mathbf{P}(\boldsymbol{\omega})$ be the finite impulse response (FIR) filter (also called the symbol of $\{\mathbf{P}_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^2}$) with its impulse response coefficients $\mathbf{P}_{\mathbf{k}}$ of the refinement equation $\boldsymbol{\Phi}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^2} \mathbf{P}_{\mathbf{k}} \boldsymbol{\Phi}(2\mathbf{x} - \mathbf{k})$ (a factor $\frac{1}{4}$ is multiplied)

$$\mathbf{P}(\boldsymbol{\omega}) = \frac{1}{4} \sum_{\mathbf{k} \in \mathbb{Z}^2} e^{-i\mathbf{k}\boldsymbol{\omega}} \mathbf{P}_{\mathbf{k}}, \quad \boldsymbol{\omega} \in \mathbb{R}^2. \quad (6.2)$$

Let $\boldsymbol{\Psi}^{(\ell)}, \tilde{\boldsymbol{\Psi}}^{(\ell)}, \ell = 1, \dots, L$ generate multiple bi-frames (bi-frames) of $L_2(\mathbb{R}^2)$ defined by

$$\boldsymbol{\Psi}^{(\ell)}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^2} \mathbf{Q}_{\mathbf{k}}^{(\ell)} \boldsymbol{\Phi}(2\mathbf{x} - \mathbf{k}), \quad \tilde{\boldsymbol{\Psi}}^{(\ell)}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^2} \tilde{\mathbf{Q}}_{\mathbf{k}}^{(\ell)} \tilde{\boldsymbol{\Phi}}(2\mathbf{x} - \mathbf{k}), \quad \mathbf{x} \in \mathbb{R}^2.$$

The symbol of $\{\mathbf{Q}_{\mathbf{k}}^{(\ell)}\}_{\mathbf{k} \in \mathbb{Z}^2}$ with its impulse response coefficients $\mathbf{Q}_{\mathbf{k}}^{(\ell)}$ of $\boldsymbol{\Psi}^{(\ell)}$ is

$$\mathbf{Q}^{(\ell)}(\boldsymbol{\omega}) = \frac{1}{4} \sum_{\mathbf{k} \in \mathbb{Z}^2} e^{-i\mathbf{k}\boldsymbol{\omega}} \mathbf{Q}_{\mathbf{k}}^{(\ell)}, \quad \boldsymbol{\omega} \in \mathbb{R}^2,$$

where the multi-indices $\mathbf{k} = (k_1, k_2)$ are the indices for the coefficients $\mathbf{Q}_{\mathbf{k}}^{(\ell)}, 1 \leq \ell \leq 4$. FIR frame multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}, \tilde{\mathbf{Q}}^{(4)}\}$ are said to be biorthogonal if they satisfy the biorthogonality conditions:

$$\tilde{\mathbf{P}}(\boldsymbol{\omega})^* \mathbf{P}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j) + \sum_{\ell=1}^4 \tilde{\mathbf{Q}}^{(\ell)}(\boldsymbol{\omega})^* \mathbf{Q}^{(\ell)}(\boldsymbol{\omega} + \pi \boldsymbol{\eta}_j) = \begin{cases} \mathbf{I}_2, & j = 0, \\ \mathbf{0}_2, & 1 \leq j \leq 3, \end{cases} \quad (6.3)$$

where

$$1 \leq \ell, \ell' \leq 4, \quad \boldsymbol{\eta}_0 = (0, 0), \quad \boldsymbol{\eta}_1 = (-1, -1), \quad \boldsymbol{\eta}_2 = (1, 0), \quad \boldsymbol{\eta}_3 = (0, 1),$$

are the representatives of the group $\mathbb{Z}^2 \setminus (2\mathbb{Z}^2)$.

Let the two multiscaling functions $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ satisfy the refinement equations

$$\hat{\boldsymbol{\Phi}}(2\boldsymbol{\omega}) = \mathbf{P}(\boldsymbol{\omega}) \hat{\boldsymbol{\Phi}}(\boldsymbol{\omega}), \quad \hat{\tilde{\boldsymbol{\Phi}}}(2\boldsymbol{\omega}) = \tilde{\mathbf{P}}(\boldsymbol{\omega}) \hat{\tilde{\boldsymbol{\Phi}}}(\boldsymbol{\omega}), \quad (6.4)$$

for some FIR filters $\mathbf{P}(\boldsymbol{\omega})$ and $\tilde{\mathbf{P}}(\boldsymbol{\omega})$ which are 2×2 matrices of 2π -periodic trigonometric polynomials, where \hat{f} denotes the Fourier transform of a function f on \mathbb{R}^2

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\mathbf{x}) e^{-i\mathbf{x}\boldsymbol{\omega}} d\mathbf{x},$$

where $\mathbf{x} \cdot \boldsymbol{\omega} = x_1 \omega_1 + x_2 \omega_2$.

Let $\boldsymbol{\Psi}^{(\ell)}$ and $\tilde{\boldsymbol{\Psi}}^{(\ell)}, \ell = 1, \dots, L$, defined by

$$\hat{\boldsymbol{\Psi}}^{(\ell)}(2\boldsymbol{\omega}) = \mathbf{Q}^{(\ell)}(\boldsymbol{\omega}) \hat{\boldsymbol{\Phi}}(\boldsymbol{\omega}), \quad \hat{\tilde{\boldsymbol{\Psi}}}^{(\ell)}(2\boldsymbol{\omega}) = \tilde{\mathbf{Q}}^{(\ell)}(\boldsymbol{\omega}) \hat{\tilde{\boldsymbol{\Phi}}}(\boldsymbol{\omega}), \quad \ell = 1, \dots, L, \quad (6.5)$$

for some FIR frame multifilters $\mathbf{Q}^{(\ell)}(\boldsymbol{\omega})$ and $\tilde{\mathbf{Q}}^{(\ell)}(\boldsymbol{\omega})$ which are some 2×2 matrices of 2π -periodic trigonometric polynomials.

In this section, we study multiple bi-frames with 4 frame generators [76]. For a multiple frame filter set $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$, with $\mathbf{Q}^{(0)}(\boldsymbol{\omega}) = \mathbf{P}(\boldsymbol{\omega})$, write $\mathbf{Q}^{(\ell)}, 1 \leq \ell \leq 4$ as

$$\mathbf{Q}^{(\ell)} = \frac{1}{2}(\mathbf{Q}_0^{(\ell)}(2\boldsymbol{\omega}) + \mathbf{Q}_1^{(\ell)}(2\boldsymbol{\omega})e^{i(\omega_1+\omega_2)} + \mathbf{Q}_2^{(\ell)}(2\boldsymbol{\omega})e^{-i\omega_1} + \mathbf{Q}_3^{(\ell)}(2\boldsymbol{\omega})e^{-i\omega_2}), \quad (6.6)$$

where $\mathbf{Q}_k^{(\ell)}(\boldsymbol{\omega}), 0 \leq k \leq 3$ are matrix trigonometric polynomials and $\boldsymbol{\omega} = (\omega_1, \omega_2)$.

The **polyphase matrix** of the frame multifilter set is $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$ to be the 10×8 matrix $\mathbf{V}(\boldsymbol{\omega})$ given by

$$\mathbf{V}(\boldsymbol{\omega}) = [\mathbf{Q}_k^{(\ell)}(\boldsymbol{\omega})]_{0 \leq \ell \leq 4, 0 \leq k \leq 3}, \quad (6.7)$$

and

$$\tilde{\mathbf{V}}(\boldsymbol{\omega}) = [\tilde{\mathbf{Q}}_k^{(\ell)}(\boldsymbol{\omega})]_{0 \leq \ell \leq 4, 0 \leq k \leq 3}. \quad (6.8)$$

$$\mathbf{V}(\boldsymbol{\omega}) := \begin{bmatrix} \mathbf{P}_0(\boldsymbol{\omega}) & \mathbf{P}_1(\boldsymbol{\omega}) & \mathbf{P}_2(\boldsymbol{\omega}) & \mathbf{P}_3(\boldsymbol{\omega}) \\ \mathbf{Q}_0^{(1)}(\boldsymbol{\omega}) & \mathbf{Q}_1^{(1)}(\boldsymbol{\omega}) & \mathbf{Q}_2^{(1)}(\boldsymbol{\omega}) & \mathbf{Q}_3^{(1)}(\boldsymbol{\omega}) \\ \mathbf{Q}_0^{(2)}(\boldsymbol{\omega}) & \mathbf{Q}_1^{(2)}(\boldsymbol{\omega}) & \mathbf{Q}_2^{(2)}(\boldsymbol{\omega}) & \mathbf{Q}_3^{(2)}(\boldsymbol{\omega}) \\ \mathbf{Q}_0^{(3)}(\boldsymbol{\omega}) & \mathbf{Q}_1^{(3)}(\boldsymbol{\omega}) & \mathbf{Q}_2^{(3)}(\boldsymbol{\omega}) & \mathbf{Q}_3^{(3)}(\boldsymbol{\omega}) \\ \mathbf{Q}_0^{(4)}(\boldsymbol{\omega}) & \mathbf{Q}_1^{(4)}(\boldsymbol{\omega}) & \mathbf{Q}_2^{(4)}(\boldsymbol{\omega}) & \mathbf{Q}_3^{(4)}(\boldsymbol{\omega}) \end{bmatrix}_{10 \times 8}, \quad (6.9)$$

$$\tilde{\mathbf{V}}(\boldsymbol{\omega}) := \begin{bmatrix} \tilde{\mathbf{P}}_0(\boldsymbol{\omega}) & \tilde{\mathbf{P}}_1(\boldsymbol{\omega}) & \tilde{\mathbf{P}}_2(\boldsymbol{\omega}) & \tilde{\mathbf{P}}_3(\boldsymbol{\omega}) \\ \tilde{\mathbf{Q}}_0^{(1)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_1^{(1)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_2^{(1)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_3^{(1)}(\boldsymbol{\omega}) \\ \tilde{\mathbf{Q}}_0^{(2)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_1^{(2)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_2^{(2)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_3^{(2)}(\boldsymbol{\omega}) \\ \tilde{\mathbf{Q}}_0^{(3)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_1^{(3)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_2^{(3)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_3^{(3)}(\boldsymbol{\omega}) \\ \tilde{\mathbf{Q}}_0^{(4)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_1^{(4)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_2^{(4)}(\boldsymbol{\omega}) & \tilde{\mathbf{Q}}_3^{(4)}(\boldsymbol{\omega}) \end{bmatrix}_{10 \times 8}. \quad (6.10)$$

Define,

$$[\mathbf{P}(\boldsymbol{\omega}), \mathbf{Q}^{(1)}(\boldsymbol{\omega}), \mathbf{Q}^{(2)}(\boldsymbol{\omega}), \mathbf{Q}^{(3)}(\boldsymbol{\omega}), \mathbf{Q}^{(4)}(\boldsymbol{\omega})]^T = \frac{1}{2}\mathbf{V}(2\boldsymbol{\omega})\mathbf{I}_0(\boldsymbol{\omega}), \quad (6.11)$$

6.2 6-Fold Symmetry Dyadic Multiple Bi-frames and Associated Templates

We know that algorithms can be given by templates to make the implementation easy. We can represent the multiresolution algorithms for regular vertices as templates. Then, we can construct template-based frame. To do this approach, let \mathcal{M}_0 denotes the regular

triangular mesh which represented from a regular infinite mesh $\mathcal{C} = \{\mathbf{c}_k\}_{k \in \mathbf{Z}^2}$ that can be showed as regular triangular mesh \mathcal{M}_0 . Here we separate the nodes of \mathcal{M}_0 , which is the regular triangular mesh, into two different groups. The first group containing the nodes with indices $(2k_1, 2k_2)$ of $2\mathbf{Z}^2$ for the coarse mesh and the remaining nodes with indices of $\mathbf{Z}^2 \setminus (2\mathbf{Z}^2)$ will be in the second group.

See Fig. 5.5, on the left, we show the regular triangular mesh \mathcal{M}_0 , with indices of \mathbf{Z}^2 , which represents the regular infinite mesh $\mathcal{C} = \{\mathbf{c}_k\}_{k \in \mathbf{Z}^2}$. The middle of the figure shows the refinement (dyadic) and the nodes with \bigcirc form the coarse mesh. On the right of the figure, the circle \bigcirc denotes the first group type V **nodes** (or **vertex nodes**) and we make type E **nodes** (or **edge nodes**) as three groups with labels in:

$$\{2\mathbf{k} - (1, 1)\}_{k \in \mathbf{Z}^2}, \{2\mathbf{k} + (1, 0)\}_{k \in \mathbf{Z}^2}, \{2\mathbf{k} + (0, 1)\}_{k \in \mathbf{Z}^2}.$$

which are denoted by \square , \triangle , and ∇ respectively.

For a regular mesh $C = \{\mathbf{c}_k\}_{k \in \mathbf{Z}^2}$ with vertices \mathbf{c}_k . Let $\{\mathbf{c}_k^1\}_k$ be the “approximation” and $\{\mathbf{d}_k^{(1, \ell)}\}_k$, $1 \leq \ell \leq 4$ be the “the details”. Associating the set of data $\{\mathbf{c}_{2k}\}_{k \in \mathbf{Z}^2}$ with type V nodes, and the three sets of the data $\{\mathbf{c}_{2k-(1,1)}\}_{k \in \mathbf{Z}^2}$, $\{\mathbf{c}_{2k+(1,0)}\}_{k \in \mathbf{Z}^2}$ and $\{\mathbf{c}_{2k+(0,1)}\}_{k \in \mathbf{Z}^2}$ associated with the previous three groups of type E nodes. Then, the multiple frame decomposition algorithms and reconstruction algorithms can be given by templates. For initial data $\{\mathbf{c}_k\}_k$, denotes

$$\mathbf{v}_k = \mathbf{c}_{2k}, \mathbf{e}_k^{(1)} = \mathbf{c}_{2k-(1,1)}, \mathbf{e}_k^{(2)} = \mathbf{c}_{2k+(1,0)}, \mathbf{e}_k^{(3)} = \mathbf{c}_{2k+(0,1)}, \mathbf{k} \in \mathbf{Z}^2. \quad (6.12)$$

Now we have four groups of data. For \mathbf{v}_k , type V are vertices. For any of $\mathbf{e}_k^{(\ell)}$, $\ell = 1, 2, 3$, we call type E vertices. Denote

$$\tilde{\mathbf{v}}_k = \mathbf{c}_k^1, \tilde{\mathbf{g}}_k = \mathbf{d}_k^{(1,1)}, \tilde{\mathbf{e}}_k^{(1)} = \mathbf{d}_k^{(1,2)}, \tilde{\mathbf{e}}_k^{(2)} = \mathbf{d}_k^{(1,3)}, \tilde{\mathbf{e}}_k^{(3)} = \mathbf{d}_k^{(1,4)}, \quad (6.13)$$

where \mathbf{v}_k , $\tilde{\mathbf{g}}_k$ and $\mathbf{e}_k^{(\ell)}$, $\ell = 1, 2, 3$ are 1×2 row-vectors.

When we use a frame multifilter bank $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$ as the analysis multifilter bank, the multiple frame multiresolution decomposition algorithm for input data \mathbf{c}_k is

$$\begin{cases} \mathbf{c}_k^1 = \frac{1}{4} \sum_{k' \in \mathbf{Z}^2} \mathbf{c}_{k'} \mathbf{P}_{k'-2k}^T, & \mathbf{d}_k^{(1,1)} = \frac{1}{4} \sum_{k' \in \mathbf{Z}^2} \mathbf{c}_{k'} \mathbf{Q}_{k'-2k}^{(1)T}, \\ \mathbf{d}_k^{(1,2)} = \frac{1}{4} \sum_{k' \in \mathbf{Z}^2} \mathbf{c}_{k'} \mathbf{Q}_{k'-2k}^{(2)T}, & \mathbf{d}_k^{(1,3)} = \frac{1}{4} \sum_{k' \in \mathbf{Z}^2} \mathbf{c}_{k'} \mathbf{Q}_{k'-2k}^{(3)T}, & \mathbf{d}_k^{(1,4)} = \frac{1}{4} \sum_{k' \in \mathbf{Z}^2} \mathbf{c}_{k'} \mathbf{Q}_{k'-2k}^{(4)T}, \end{cases} \quad (6.14)$$

for $\mathbf{k}, \mathbf{k}' \in \mathbf{Z}^2$, where \mathbf{c}_k^1 , $\mathbf{d}_k^{(1,1)}$, $\mathbf{d}_k^{(1,2)}$, $\mathbf{d}_k^{(1,3)}$ and $\mathbf{d}_k^{(1,4)}$ are 1×2 “row-vectors” and the masks $\mathbf{P}_{k'-2k}$ and $\mathbf{Q}_{k'-2k}^{(\ell)}$, $\ell = 1, \dots, 4$, are 2×2 square matrices. If the synthesis multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}, \tilde{\mathbf{Q}}^{(4)}\}$ is biorthogonal to $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$, then the input

data $\{\mathbf{c}_k\}$ can be recovered from \mathbf{c}_k^1 , $\mathbf{d}_k^{(1,1)}$, $\mathbf{d}_k^{(1,2)}$, $\mathbf{d}_k^{(1,3)}$ and $\mathbf{d}_k^{(1,4)}$ by the multiple frame multiresolution reconstruction algorithm:

$$\mathbf{c}'_k = \sum_{\mathbf{k}' \in \mathbf{Z}^2} \{\mathbf{c}_{\mathbf{k}'}^1 \tilde{\mathbf{P}}_{\mathbf{k}-2\mathbf{k}'} + \mathbf{d}_{\mathbf{k}'}^{(1,1)} \tilde{\mathbf{Q}}_{\mathbf{k}-2\mathbf{k}'}^{(1)} + \mathbf{d}_{\mathbf{k}'}^{(1,2)} \tilde{\mathbf{Q}}_{\mathbf{k}-2\mathbf{k}'}^{(2)} + \mathbf{d}_{\mathbf{k}'}^{(1,3)} \tilde{\mathbf{Q}}_{\mathbf{k}-2\mathbf{k}'}^{(3)} + \mathbf{d}_{\mathbf{k}'}^{(1,4)} \tilde{\mathbf{Q}}_{\mathbf{k}-2\mathbf{k}'}^{(4)}\}, \quad (6.15)$$

where $\mathbf{k} \in \mathbf{Z}^2$ and the frame multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}, \tilde{\mathbf{Q}}^{(4)}\}$ is the synthesis multifilter bank. $\{\tilde{\mathbf{v}}_k\}_k$ is called the lowpass output and $\{\mathbf{d}_k^{(1,1)}\}_k$, $\{\mathbf{d}_k^{(1,2)}\}_k$, $\{\mathbf{d}_k^{(1,3)}\}_k$, $\{\mathbf{d}_k^{(1,4)}\}_k$ are called the highpass outputs of $\{\mathbf{c}_k\}_k$ respectively.

Theorem 4. *If $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}, \tilde{\mathbf{Q}}^{(4)}\}$ are biorthogonal frame multifilter banks, then the input data $\{\mathbf{c}_k\}$ can be recovered from its approximation $\{\mathbf{c}_k^1\}$ and details $\{\mathbf{d}_k^{(1,1)}\}$, $\{\mathbf{d}_k^{(1,2)}\}$, $\{\mathbf{d}_k^{(1,3)}\}$, $\{\mathbf{d}_k^{(1,4)}\}$, namely the input data $\{\mathbf{c}_k\}$ is exactly $\{\mathbf{c}'_k\}$.*

Proof: Let $\mathbf{c}^1(2\omega)$, $\mathbf{d}^{(1,1)}(2\omega)$, $\mathbf{d}^{(1,2)}(2\omega)$, $\mathbf{d}^{(1,3)}(2\omega)$, $\mathbf{d}^{(1,4)}(2\omega)$, $\mathbf{c}'(\omega)$ denote the Z-transforms of \mathbf{c}_k^1 , $\mathbf{d}_k^{(1,1)}$, $\mathbf{d}_k^{(1,2)}$, $\mathbf{d}_k^{(1,3)}$, $\mathbf{d}_k^{(1,4)}$, and \mathbf{c}'_k , respectively, where $\mathbf{k} \in \mathbf{Z}^2$. The Z-transforms $\mathbf{c}(\omega)$ of $\mathbf{c}_k = \{\mathbf{c}_k\}_k$ is $\mathbf{c}(\omega) = \frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^2} \mathbf{c}_k e^{-i\mathbf{k}\omega}$. In the frequency domain, we have

$$\mathbf{c}^1(2\omega) = \frac{1}{4} \sum_{j=0}^3 \mathbf{c}(\omega + \pi\eta_j) \overline{\mathbf{P}(\omega + \pi\eta_j)}^T,$$

$$\mathbf{d}^{(1,1)}(2\omega) = \frac{1}{4} \sum_{j=0}^3 \mathbf{c}(\omega + \pi\eta_j) \overline{\mathbf{Q}^{(1)}(\omega + \pi\eta_j)}^T,$$

$$\mathbf{d}^{(1,2)}(2\omega) = \frac{1}{4} \sum_{j=0}^3 \mathbf{c}(\omega + \pi\eta_j) \overline{\mathbf{Q}^{(2)}(\omega + \pi\eta_j)}^T,$$

$$\mathbf{d}^{(1,3)}(2\omega) = \frac{1}{4} \sum_{j=0}^3 \mathbf{c}(\omega + \pi\eta_j) \overline{\mathbf{Q}^{(3)}(\omega + \pi\eta_j)}^T,$$

$$\mathbf{d}^{(1,4)}(2\omega) = \frac{1}{4} \sum_{j=0}^3 \mathbf{c}(\omega + \pi\eta_j) \overline{\mathbf{Q}^{(4)}(\omega + \pi\eta_j)}^T,$$

and

$$\begin{aligned} \mathbf{c}'(\omega) = & 4 \left(\mathbf{c}^1(2\omega) \tilde{\mathbf{P}}(\omega) + \mathbf{d}^{(1,1)}(2\omega) \tilde{\mathbf{Q}}^{(1)}(\omega) + \mathbf{d}^{(1,2)}(2\omega) \tilde{\mathbf{Q}}^{(2)}(\omega) \right. \\ & \left. + \mathbf{d}^{(1,3)}(2\omega) \tilde{\mathbf{Q}}^{(3)}(\omega) + \mathbf{d}^{(1,4)}(2\omega) \tilde{\mathbf{Q}}^{(4)}(\omega) \right). \end{aligned}$$

Suppose that the frame multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}, \tilde{\mathbf{Q}}^{(4)}\}$ are biorthogonal. By plugging $\mathbf{c}^1(2\omega), \mathbf{d}^{(1,1)}(2\omega), \mathbf{d}^{(1,2)}(2\omega), \mathbf{d}^{(1,3)}(2\omega)$ and $\mathbf{d}^{(1,4)}(2\omega)$ into $\mathbf{c}'(\omega)$ and if the conditions in (6.3) hold. Then, $\mathbf{c}'(\omega) = \mathbf{c}(\omega)$. Hence, the biorthogonality of $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}, \tilde{\mathbf{Q}}^{(4)}\}$ implies that the original data can be recovered from the lowpass and highpass outputs by the multiple frame reconstruction algorithms.

As in the paper [76], we consider $\mathbf{c}_{\mathbf{k}}$ with \mathbf{k} in four different cases: $(2j_1, 2j_2)$, $(2j_1 - 1, 2j_2 - 1)$, $(2j_1 + 1, 2j_2)$, $(2j_1, 2j_2 + 1)$, and with the definitions of $\mathbf{v}_{\mathbf{k}}, \mathbf{e}_{\mathbf{k}}^{(\ell)}$, $\ell = 1, 2, 3$, we can write the above reconstruction algorithm as

$$\left\{ \begin{array}{l} \mathbf{v}_{\mathbf{k}} = \sum_{\mathbf{n} \in \mathbb{Z}^2} \{ \tilde{\mathbf{v}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{P}}_{2\mathbf{n}} + \tilde{\mathbf{g}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{Q}}_{2\mathbf{n}}^{(1)} + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(1)} \tilde{\mathbf{Q}}_{2\mathbf{n}}^{(2)} + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(2)} \tilde{\mathbf{Q}}_{2\mathbf{n}}^{(3)} + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(3)} \tilde{\mathbf{Q}}_{2\mathbf{n}}^{(4)} \}, \\ \mathbf{e}_{\mathbf{k}}^{(1)} = \sum_{\mathbf{n} \in \mathbb{Z}^2} \{ \tilde{\mathbf{v}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{P}}_{2\mathbf{n}-(1,1)} + \tilde{\mathbf{g}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{Q}}_{2\mathbf{n}-(1,1)}^{(1)} + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(1)} \tilde{\mathbf{Q}}_{2\mathbf{n}-(1,1)}^{(2)} + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(2)} \tilde{\mathbf{Q}}_{2\mathbf{n}-(1,1)}^{(3)} \\ + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(3)} \tilde{\mathbf{Q}}_{2\mathbf{n}-(1,1)}^{(4)} \}, \\ \mathbf{e}_{\mathbf{k}}^{(2)} = \sum_{\mathbf{n} \in \mathbb{Z}^2} \{ \tilde{\mathbf{v}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{P}}_{2\mathbf{n}+(1,0)} + \tilde{\mathbf{g}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{Q}}_{2\mathbf{n}+(1,0)}^{(1)} + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(1)} \tilde{\mathbf{Q}}_{2\mathbf{n}+(1,0)}^{(2)} + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(2)} \tilde{\mathbf{Q}}_{2\mathbf{n}+(1,0)}^{(3)} \\ + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(3)} \tilde{\mathbf{Q}}_{2\mathbf{n}+(1,0)}^{(4)} \}, \\ \mathbf{e}_{\mathbf{k}}^{(3)} = \sum_{\mathbf{n} \in \mathbb{Z}^2} \{ \tilde{\mathbf{v}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{P}}_{2\mathbf{n}+(0,1)} + \tilde{\mathbf{g}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{Q}}_{2\mathbf{n}+(0,1)}^{(1)} + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(1)} \tilde{\mathbf{Q}}_{2\mathbf{n}+(0,1)}^{(2)} + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(2)} \tilde{\mathbf{Q}}_{2\mathbf{n}+(0,1)}^{(3)} \\ + \tilde{\mathbf{e}}_{\mathbf{k}-\mathbf{n}}^{(3)} \tilde{\mathbf{Q}}_{2\mathbf{n}+(0,1)}^{(4)} \}. \end{array} \right. \quad (6.16)$$

As in [76], we observe that when we set the “details“ $\tilde{\mathbf{g}}_{\mathbf{k}}, \tilde{\mathbf{e}}_{\mathbf{k}}^{(\ell)}$, $\ell = 1, 2, 3$ to be 1×2 zero row vectors, the reconstruction algorithm (6.16) reduced to the subdivision algorithm:

$$\left\{ \begin{array}{l} \mathbf{v}_{\mathbf{k}} = \sum_{\mathbf{n} \in \mathbb{Z}^2} \tilde{\mathbf{v}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{P}}_{2\mathbf{n}}, \\ \mathbf{e}_{\mathbf{k}}^{(1)} = \sum_{\mathbf{n} \in \mathbb{Z}^2} \tilde{\mathbf{v}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{P}}_{2\mathbf{n}-(1,1)}, \\ \mathbf{e}_{\mathbf{k}}^{(2)} = \sum_{\mathbf{n} \in \mathbb{Z}^2} \tilde{\mathbf{v}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{P}}_{2\mathbf{n}+(1,0)}, \\ \mathbf{e}_{\mathbf{k}}^{(3)} = \sum_{\mathbf{n} \in \mathbb{Z}^2} \tilde{\mathbf{v}}_{\mathbf{k}-\mathbf{n}} \tilde{\mathbf{P}}_{2\mathbf{n}+(0,1)}, \end{array} \right. \quad (6.17)$$

from the above algorithm we can derive the templates.

The multiresolution algorithms can be given by templates by associating the outputs $\tilde{\mathbf{v}}_{\mathbf{k}}, \tilde{\mathbf{g}}_{\mathbf{k}}, \tilde{\mathbf{e}}_{\mathbf{k}}^{(\ell)}$, $\ell = 1, 2, 3$ with the node of M_0 , which is the regular triangular mesh. So, we will associate the lowpass output $\{\tilde{\mathbf{v}}_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^2}$ and the first highpass output $\tilde{\mathbf{g}}_{\mathbf{k}}$ with type V nodes with $(2k_1, 2k_2)$ and associate the highpass outputs $\tilde{\mathbf{e}}_{\mathbf{k}}^{(\ell)}$, $\ell = 1, 2, 3$ with the type E nodes $(k_1 - 1, k_2 - 1)$, $(k_1 + 1, k_2)$ and $(k_1, k_2 + 1)$, respectively.

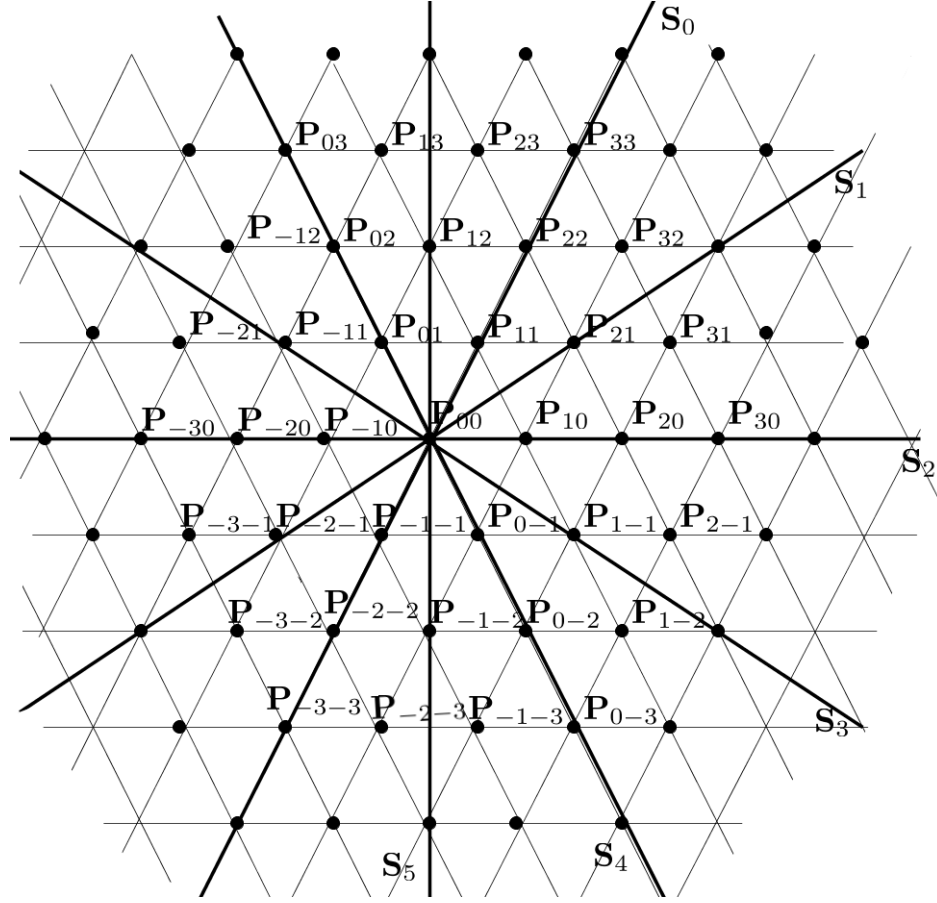


Figure 6.1: Symmetry lines for the lowpass multifilter bank \mathbf{P}_{k_1, k_2}

From [76], analysis and synthesis algorithm templates must have certain symmetry when they are used for surface processing. First, since all the highpass outputs $\tilde{\mathbf{e}}_{\mathbf{k}}^{(\ell)}$, $\ell = 1, 2, 3$ are associated with the type E vertices, then the templates to obtain $\tilde{\mathbf{e}}_{\mathbf{k}}^{(\ell)}$, $\ell = 1, 2, 3$ must be the same and we should treat them equally. Also, when we recover $\mathbf{e}_{\mathbf{k}}^{(\ell)}$, $\ell = 1, 2, 3$, we should use identical templates because they all are associated with the type E vertices. So, the templates to obtain $\tilde{\mathbf{e}}_{\mathbf{k}}^{(\ell)}$, $\ell = 1, 2, 3$ (we use the same templates to obtain all of them) and the templates to recover $\mathbf{e}_{\mathbf{k}}^{(\ell)}$, $\ell = 1, 2, 3$ (we use the same templates to recover all of them) have certain symmetry. Moreover, in order to obtain $\tilde{\mathbf{v}}_{\mathbf{k}}$ and the first highpass output $\tilde{\mathbf{g}}_{\mathbf{k}}$, which are associated with the type V nodes, and to recover $\mathbf{v}_{\mathbf{k}}$ we must use rotational and reflective invariant templates for the coarse mesh.

The 6-fold line symmetry of biorthogonal wavelet multifilter banks in the scalar case is introduced in [76, 66]. To construct 6-fold symmetric multiple bi-frames, first, we represent the decomposition and reconstruction algorithms by symmetric templates by using the idea of the lifting scheme where the algorithm templates are given by many repeated steps and every step in the algorithm is given by a template. Then, we can obtain the corresponding

multiple bi-frame multifilter bank. After solving the linear equation systems, we find some parameters and then we choose the parameters that give us optimal smoothness order and vanishing moments of the multiwavelet or multiple frames. In the following subsections, consider the algorithms given by 2 and 3 steps and we describe multiresolution algorithms by using \mathbf{v}, \mathbf{e} and $\tilde{\mathbf{v}}, \tilde{\mathbf{g}}, \tilde{\mathbf{e}}$.

We will introduce the definition of the 6-fold line symmetry of biorthogonal wavelet

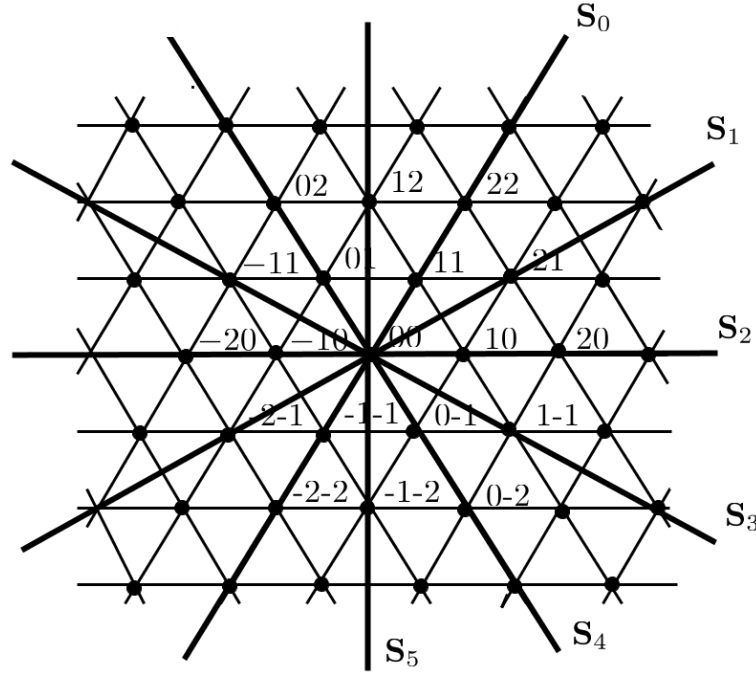


Figure 6.2: Symmetry axes for frame lowpass filter p and the first highpass filter $q^{(1)}$.

multifilter banks in the scalar case from [76, 66].

A (dyadic) frame filter set $\{p, q^{(1)}, \dots, q^{(4)}\}$ is said to have 6-fold axial symmetry or a full set of symmetries if

- (i) coefficients $p_{\mathbf{k}}$ and $q_{\mathbf{k}}^{(1)}$ of its lowpass filter $p(\omega)$ and first highpass filter $q^{(1)}(\omega)$ are symmetric around axes S_0, \dots, S_5 on the left of Fig. 6.2;
- (ii) the coefficients $q_{\mathbf{k}}^{(2)}$ of its second highpass filter is symmetric around the axes S_0, S_3'' on the right of Fig. 6.3;
- (iii) $q_{\mathbf{k}}^{(3)}, q_{\mathbf{k}}^{(4)}$ of other two highpass filters $q^{(3)}(\omega)$ and $q^{(4)}(\omega)$ are the $\frac{2\pi}{3}$ and the $\frac{4\pi}{3}$ rotations of $q_{\mathbf{k}}^{(2)}$.

We will introduce the following propositions about the relation between 6-fold symmetry and the polyphase matrix $\mathbf{V}(\omega)$ [76] .

Proposition 1. *A frame filter set $\{p, q^{(1)}, \dots, q^{(4)}\}$ has 6-fold symmetry if and only if its polyphase matrix $\mathbf{V}(\omega)$ satisfies*

$$\mathbf{V}(\mathbf{L}_0 \omega) = \mathbf{S}_{01} \mathbf{V}(\omega) \mathbf{S}_{02}, \quad \mathbf{V}(\mathbf{R}_1^{-T} \omega) = \mathbf{S}_1(\omega) \mathbf{V}(\omega) \mathbf{S}_2(\omega),$$

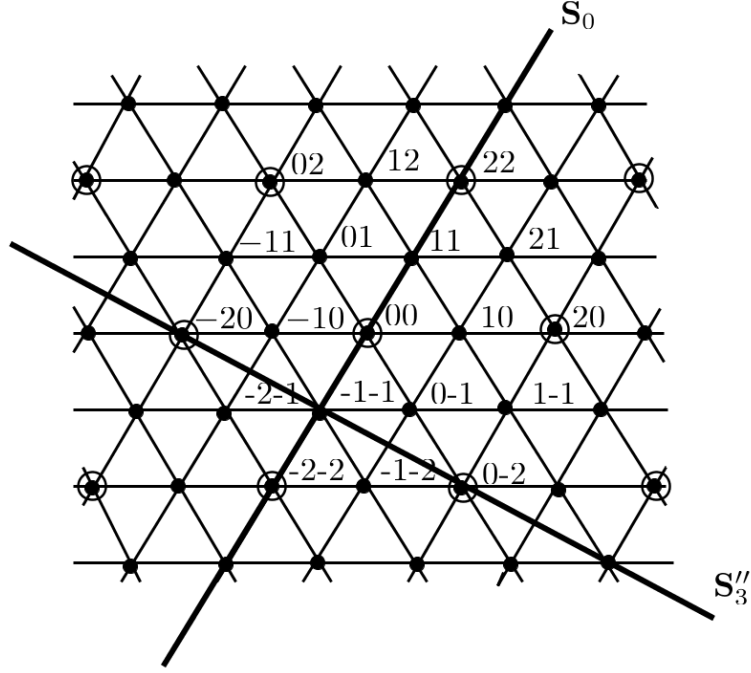


Figure 6.3: Symmetry axes for the second frame highpass filter $q^{(2)}$.

where

$$\mathbf{R}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \mathbf{L}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{S}_{01} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_0 \end{bmatrix}, \mathbf{S}_{02} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_0 \end{bmatrix},$$

$$\mathbf{S}_1(\boldsymbol{\omega}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\omega_2} \\ 0 & 0 & e^{-i(\omega_1+\omega_2)} & 0 & 0 \\ 0 & 0 & 0 & e^{i\omega_1} & 0 \end{bmatrix},$$

$$\mathbf{S}_2(\boldsymbol{\omega}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{i(\omega_1+\omega_2)} & 0 \\ 0 & 0 & 0 & e^{-i\omega_1} \\ 0 & e^{-i\omega_2} & 0 & 0 \end{bmatrix}.$$

where \mathbf{I}_3 and \mathbf{I}_2 are the identity matrices and $\mathbf{0}$ is the zero matrix.

6.2.1 2-Step Multiple Bi-frame Multiresolution Algorithm

In this subsection we consider a 2-step multiple bi-frame algorithm. For given 1×2 row vectors $\{\mathbf{v}\}$ and $\{\mathbf{e}\}$ (or given triangular mesh C as introduced in the beginning of this section), Fig. 6.4 shows the multiresolution decomposition algorithm that is given by (6.18)-(6.19), where \mathbf{B} , \mathbf{M} , \mathbf{N} , \mathbf{A} , \mathbf{C} , \mathbf{H} , \mathbf{J} , and \mathbf{T} are 2×2 matrices and the entries of these matrices are some constants in \mathbb{R} to be determined.

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \mathbf{N} = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}, \mathbf{J} = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}, \text{ and } \mathbf{T} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}.$$

2-step Decomposition Algorithm:

$$\text{Step 1. } \begin{cases} \tilde{\mathbf{v}} = \{\mathbf{v} - (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) \mathbf{M}\} \mathbf{B}^{-1}, \\ \tilde{\mathbf{g}} = \mathbf{v} - (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) \mathbf{N}; \end{cases} \quad (6.18)$$

$$\text{Step 2. } \tilde{\mathbf{e}} = \mathbf{e} - (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1) \mathbf{A} - (\tilde{\mathbf{v}}_2 + \tilde{\mathbf{v}}_3) \mathbf{C} - (\tilde{\mathbf{g}}_0 + \tilde{\mathbf{g}}_1) \mathbf{H} - (\tilde{\mathbf{g}}_2 + \tilde{\mathbf{g}}_3) \mathbf{J}. \quad (6.19)$$

We obtain lowpass output $\tilde{\mathbf{v}}$ and the first highpass output $\tilde{\mathbf{g}}$, which both are associated with type V nodes of \mathcal{M}_0 . Then, in Step 2, we use the obtained $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{g}}$ to obtain other three highpass outputs $\{\tilde{\mathbf{e}}\} (= \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(1)}\} \cup \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(2)}\} \cup \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(3)}\})$ associated with type E nodes of \mathcal{M}_0 .

The reconstruction algorithm is the backward algorithm of the decomposition algorithm as shown in Fig. 6.5. We give the multiresolution reconstruction algorithm, where \mathbf{B} , \mathbf{M} , \mathbf{N} , \mathbf{A} , \mathbf{C} , \mathbf{H} , \mathbf{J} , and \mathbf{T} are the same 2×2 matrices in the decomposition algorithm.

2-step Reconstruction Algorithm:

$$\text{Step 1. } \mathbf{e} = \tilde{\mathbf{e}} + (\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1) \mathbf{A} + (\tilde{\mathbf{v}}_2 + \tilde{\mathbf{v}}_3) \mathbf{C} + (\tilde{\mathbf{g}}_0 + \tilde{\mathbf{g}}_1) \mathbf{H} + (\tilde{\mathbf{g}}_2 + \tilde{\mathbf{g}}_3) \mathbf{J}; \quad (6.20)$$

$$\text{Step 2. } \mathbf{v} = \{\tilde{\mathbf{v}} \mathbf{B} + (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) \mathbf{M}\} \mathbf{T} \\ + \{\tilde{\mathbf{g}} + (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) \mathbf{N}\} (\mathbf{I}_2 - \mathbf{T}). \quad (6.21)$$

First in Step 1, we replace $\tilde{\mathbf{e}}$ by \mathbf{e} . Then in Step 2 with \mathbf{e} obtained in Step 1, we replace $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{g}}$ by \mathbf{v} . So, the inputs here are $\tilde{\mathbf{v}}$, $\tilde{\mathbf{g}}$, $\tilde{\mathbf{e}}$ and the outputs are \mathbf{v} , \mathbf{e} .

We can follow the detailed calculations in [73], for 1-D filters to obtain the multifilter bank $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}, \tilde{\mathbf{Q}}^{(4)}\}$ corresponding to this 2-D (2-step) algorithm. Let $\boldsymbol{\omega} = (\omega_1, \omega_2)$. We use x and y to denote $e^{-i\omega_1}$, $e^{-i\omega_2}$, respectively:

$$x = e^{-i\omega_1}, \quad y = e^{-i\omega_2}.$$

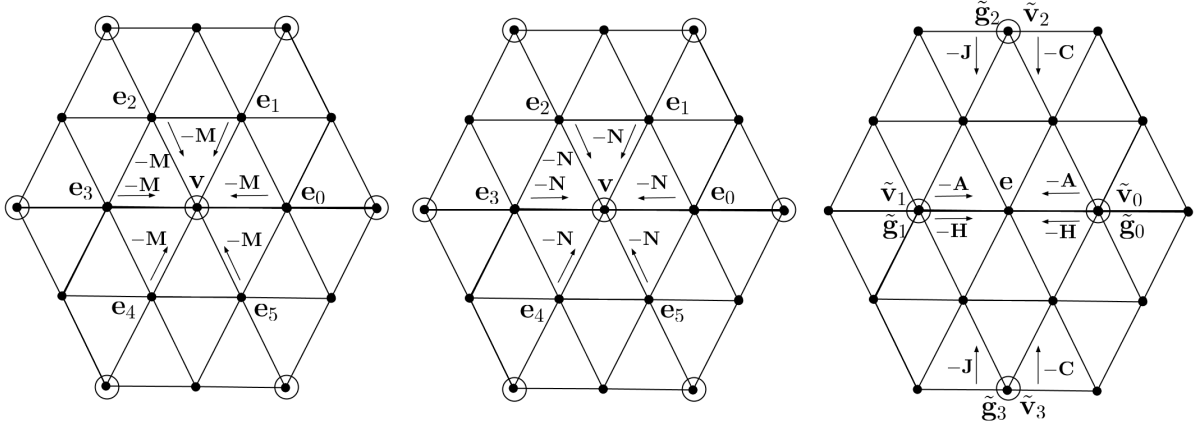


Figure 6.4: Left and middle: Decomposition Algorithm Step 1 (left: template to get lowpass output $\tilde{\mathbf{v}}$ and middle: template to obtain first highpass output $\tilde{\mathbf{g}}$ which similar with $-\mathbf{M}$ replaced by $-\mathbf{N}$); Right: Decomposition Algorithm Step 2.

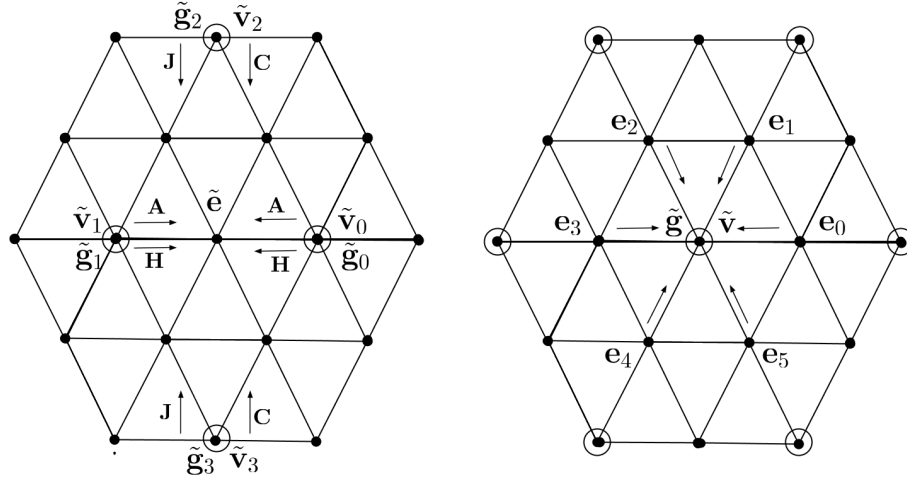


Figure 6.5: Left: Reconstruction Algorithm Step 1; Right: Reconstruction Algorithm Step 2.

We derive the analysis lowpass multifilter \mathbf{P}_{k_1, k_2} which is satisfy the sum rule of order 1, namely:

$$\begin{aligned} \mathbf{P}_{0,0} &= 4 (\mathbf{B}^{-1})^T, \\ \mathbf{P}_{1,0} &= \mathbf{P}_{0,1} = \mathbf{P}_{0,-1} = \mathbf{P}_{-1,0} = \mathbf{P}_{-1,-1} = \mathbf{P}_{1,1} = -4 (\mathbf{B}^{-1})^T \mathbf{M}^T. \end{aligned}$$

and we derive the synthesis lowpass multifilter $\tilde{\mathbf{P}}_{k_1, k_2}$ which is satisfy the sum rule of order

1, namely:

$$\begin{aligned}
\tilde{\mathbf{P}}_{0,0} &= \mathbf{B}\mathbf{T} + 6 (\mathbf{A}\mathbf{M}\mathbf{T} + \mathbf{A}\mathbf{N}(\mathbf{I}_2 - \mathbf{T})), \\
\tilde{\mathbf{P}}_{1,0} &= \tilde{\mathbf{P}}_{0,1} = \tilde{\mathbf{P}}_{0,-1} = \tilde{\mathbf{P}}_{-1,0} = \tilde{\mathbf{P}}_{-1,-1} = \tilde{\mathbf{P}}_{1,1} = \mathbf{A}, \\
\tilde{\mathbf{P}}_{2,0} &= \tilde{\mathbf{P}}_{0,2} = \tilde{\mathbf{P}}_{0,-2} = \tilde{\mathbf{P}}_{-2,0} = \tilde{\mathbf{P}}_{-2,-2} = \tilde{\mathbf{P}}_{2,2} = \mathbf{A}\mathbf{M}\mathbf{T} \\
&\quad + \mathbf{A}\mathbf{N}(\mathbf{I}_2 - \mathbf{T}) + 2 (\mathbf{C}\mathbf{M}\mathbf{T} + \mathbf{C}\mathbf{N}(\mathbf{I}_2 - \mathbf{T})), \\
\tilde{\mathbf{P}}_{1,2} &= \tilde{\mathbf{P}}_{2,1} = \tilde{\mathbf{P}}_{-1,1} = \tilde{\mathbf{P}}_{1,-1} = \tilde{\mathbf{P}}_{-1,-2} = \tilde{\mathbf{P}}_{-2,-1} = \mathbf{C}.
\end{aligned}$$

Denote

$$\left\{ \begin{aligned} \mathbf{G}_1(\boldsymbol{\omega}) &= \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -(1 + \frac{1}{xy})\mathbf{A}^T - (\frac{1}{x} + \frac{1}{y})\mathbf{C}^T & -(1 + \frac{1}{xy})\mathbf{H}^T - (\frac{1}{x} + \frac{1}{y})\mathbf{J}^T & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -(1 + x)\mathbf{A}^T - (xy + \frac{1}{y})\mathbf{C}^T & -(1 + x)\mathbf{H}^T - (xy + \frac{1}{y})\mathbf{J}^T & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ -(1 + y)\mathbf{A}^T - (xy + \frac{1}{x})\mathbf{C}^T & -(1 + y)\mathbf{H}^T - (xy + \frac{1}{x})\mathbf{J}^T & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{10 \times 10}, \\ \\ \mathbf{G}_0(\boldsymbol{\omega}) &= \begin{bmatrix} (\mathbf{B}^{-1})^T & -(1 + xy)(\mathbf{B}^{-1})^T\mathbf{M}^T & -(1 + \frac{1}{x})(\mathbf{B}^{-1})^T\mathbf{M}^T & -(1 + \frac{1}{y})(\mathbf{B}^{-1})^T\mathbf{M}^T \\ \mathbf{I}_2 & -(1 + xy)\mathbf{N}^T & -(1 + \frac{1}{x})\mathbf{N}^T & -(1 + \frac{1}{y})\mathbf{N}^T \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{10 \times 8}. \end{aligned} \right. \quad (6.22)$$

and the inverse of $\mathbf{G}(\boldsymbol{\omega})$ is a matrix whose entries are also polynomials of x, y . More precisely, $\tilde{\mathbf{G}}_1(\boldsymbol{\omega}) = (\mathbf{G}_1(\boldsymbol{\omega})^{-1})^*$ and $\tilde{\mathbf{G}}_0(\boldsymbol{\omega}) = (\mathbf{G}_0(\boldsymbol{\omega})^{-1})^*$ are given by

$$\left\{ \begin{array}{l} \tilde{\mathbf{G}}_1(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & (1+xy)\mathbf{A} + (x+y)\mathbf{C} & (1+\frac{1}{x})\mathbf{A} + (\frac{1}{xy}+y)\mathbf{C} & (1+\frac{1}{y})\mathbf{A} + (\frac{1}{xy}+x)\mathbf{C} \\ \mathbf{0}_2 & \mathbf{I}_2 & (1+xy)\mathbf{H} + (x+y)\mathbf{J} & (1+\frac{1}{x})\mathbf{H} + (\frac{1}{xy}+y)\mathbf{J} & (1+\frac{1}{y})\mathbf{H} + (\frac{1}{xy}+x)\mathbf{J} \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{10 \times 10}, \\ \\ \tilde{\mathbf{G}}_0(\omega) = \begin{bmatrix} \mathbf{BT} & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (\mathbf{I}_2 - \mathbf{T}) & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1+\frac{1}{xy})(\mathbf{MT} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T})) & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1+x)(\mathbf{MT} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T})) & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ (1+y)(\mathbf{MT} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T})) & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}_{10 \times 8}. \end{array} \right. \quad (6.23)$$

The analysis multifilter bank $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$ and the synthesis multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}, \tilde{\mathbf{Q}}^{(4)}\}$ can be written as the following:

$$[\mathbf{P}(\omega), \mathbf{Q}^{(1)}(\omega), \mathbf{Q}^{(2)}(\omega), \mathbf{Q}^{(3)}(\omega), \mathbf{Q}^{(4)}(\omega)]^T = \mathbf{G}_1(2\omega) \mathbf{G}_0(2\omega) \mathbf{I}_0(\omega), \quad (6.24)$$

$$[\tilde{\mathbf{P}}(\omega), \tilde{\mathbf{Q}}^{(1)}(\omega), \tilde{\mathbf{Q}}^{(2)}(\omega), \tilde{\mathbf{Q}}^{(3)}(\omega), \tilde{\mathbf{Q}}^{(4)}(\omega)]^T = \frac{1}{4} \tilde{\mathbf{G}}_1(2\omega) \tilde{\mathbf{G}}_0(2\omega) \mathbf{I}_0(\omega), \quad (6.25)$$

We can show that the multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}, \tilde{\mathbf{Q}}^{(4)}\}$ are biorthogonal by verifying that $\tilde{\mathbf{G}}_0^*(\omega) \mathbf{G}_0(\omega) = \mathbf{I}_8$, $\tilde{\mathbf{G}}_1^*(\omega) \mathbf{G}_1(\omega) = \mathbf{I}_{10}$, $\omega \in \mathbb{R}^2$ and we obtain that the polyphase matrices of these multifilter banks satisfy $\tilde{\mathbf{V}}(\omega)^* \mathbf{V}(\omega) = \mathbf{I}_8$, $\omega \in \mathbb{R}^2$.

Next step, we solve the system of equations for a sum rule of order one of \mathbf{P} and $\tilde{\mathbf{P}}$ and for a vanishing moment of order one for $\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}$ to obtain the parameters.

We have 19 free parameters which are b_{22} , c_{12} , c_{21} , h_{11} , h_{21} , j_{11} , j_{21} , m_{12} , n_{22} , t_{11} , t_{12} , t_{21} , t_{22} , a_{22} , c_{22} , h_{12} , h_{22} , j_{12} , j_{22} , and the other parameters are

$$\begin{aligned} a_{11} &= 0, \quad a_{12} = -c_{12}, \quad a_{21} = 0, \quad b_{11} = 4, \quad b_{12} = -6m_{12}, \quad b_{21} = 0, \quad c_{11} = -\frac{1}{2}, \quad m_{11} = -\frac{1}{2}, \\ m_{21} &= 0, \quad m_{22} = -\frac{1}{6}(6b_{22}h_{21}n_{22} + 6b_{22}j_{21}n_{22} - b_{22}h_{21} - b_{22}j_{21} - c_{21}), \quad n_{11} = -\frac{1}{6}, \\ n_{12} &= 0, \quad n_{21} = 0. \end{aligned}$$

If we choose

$$\begin{aligned} &[b_{22}, c_{12}, c_{21}, h_{11}, h_{21}, j_{11}, j_{21}, m_{12}, n_{22}, t_{11}, t_{12}, t_{21}, t_{22}, a_{22}, c_{22}, h_{12}, h_{22}, j_{12}, \\ &j_{22}] = \left[\frac{1125}{377}, \frac{1}{88}, \frac{-4687}{268}, \frac{316}{361}, \frac{15}{236}, \frac{269}{520}, \frac{16}{95}, \frac{-67}{159}, \frac{-1277}{210}, \frac{-219}{161}, \frac{-47}{80}, \frac{387}{211}, \frac{137}{154}, \frac{44}{119}, \frac{-239}{597}, \right. \\ &\left. \frac{81}{103}, \frac{697}{863}, \frac{977}{760}, \frac{-11}{167} \right]. \end{aligned}$$

By using the smoothness formula, we can obtain $\tilde{\Phi} \in W^{1.3839}$ and $\Phi \in W^{0.5441}$.
The lowpass analysis mask \mathbf{P} and the lowpass synthesis mask $\tilde{\mathbf{P}}$ will be as follows:

$$\mathbf{P}(\omega) = \begin{bmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} P_{11}(\omega) &= (0.12500000000649x^2y^2 + 0.1250000000064x^2y + 0.1250000000064xy^2 \\ &\quad + 0.250000000008xy + 0.1250000000064x + 0.1250000000064y \\ &\quad + 0.1250000000064)/(xy), \\ P_{12}(\omega) &= 0, \\ P_{21}(\omega) &= (0.03530258554692x^2y^2 + 0.03530258554692x^2y + 0.03530258554692xy^2 \\ &\quad - 0.21181551370041xy + 0.03530258554692x + 0.03530258554692y \\ &\quad + 0.03530258554692)/(xy), \\ P_{22}(\omega) &= (0.02701972954743x^2y^2 + 0.02701972954743x^2y + 0.02701972954743xy^2 + 4xy \\ &\quad + 0.02701972954743x + 0.02701972954743y + 0.02701972954743)/(xy). \end{aligned}$$

and

$$\tilde{\mathbf{P}}(\omega) = \begin{bmatrix} \tilde{P}_{11}(\omega) & \tilde{P}_{12}(\omega) \\ \tilde{P}_{21}(\omega) & \tilde{P}_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} \tilde{P}_{11}(\omega) &= (0.106424472725x^4y^4 + x^4y^3 + x^3y^4 + 0.106424472725x^4y^2 \\ &\quad + 0.106424472725x^2y^4 + x^3y - 0.388535034x^2y^2 + xy^3 + 0.106424472725x^2 \\ &\quad + 0.106424472725y^2 + x + y + 0.106424472725)/(x^2y^2), \\ \tilde{P}_{12}(\omega) &= (0.00208953590475x^4y^4 + 0.00284090909x^4y^3 + 0.00284090909x^3y^4 \\ &\quad + 0.00208953590475x^4y^2 - 0.00284090909x^3y^3 + 0.00208953590475x^2y^4 \\ &\quad - 0.00284090909x^3y^2 - 0.00284090909x^2y^3 + 0.00284090909x^3y - 0.0125342777775x^2y^2 \\ &\quad + 0.00284090909xy^3 - 0.00284090909x^2y - 0.00284090909xy^2 + 0.00208953590475x^2 \\ &\quad - 0.0284090909xy + 0.00208953590475y^2 + 0.00284090909x + 0.00284090909y \\ &\quad + 0.00208953590475)/(x^2y^2), \\ \tilde{P}_{21}(\omega) &= (-3.8145039475x^4y^4 - 4.3722014925x^4y^3 - 4.3722014925x^3y^4 \\ &\quad - 3.8145039475x^4y^2 - 3.8145039475x^2y^4 - 4.3722014925x^3y + 7.4720925125x^2y^2 \\ &\quad - 4.3722014925xy^3 - 3.8145039475x^2 - 3.8145039475y^2 - 4.3722014925x \\ &\quad - 4.3722014925y - 3.8145039475)/(x^2y^2), \end{aligned}$$

$$\begin{aligned}\tilde{P}_{22}(\omega) = & (-0.0668737559x^4y^4 - 0.0100837521x^4y^3 - 0.0100837521x^3y^4 \\ & -0.0668737559x^4y^2 + 0.0924369748x^3y^3 - 0.0668737559x^2y^4 + 0.0924369748x^3y^2 \\ & +0.0924369748x^2y^3 - 0.0100837521x^3y + 0.2515828038x^2y^2 - 0.0100837521xy^3 \\ & +0.0924369748x^2y + 0.0924369748xy^2 - 0.0668737559x^2 + 0.0924369748xy \\ & -0.0668737559y^2 - 0.0100837521x - 0.0100837521y - 0.0668737559)/(x^2y^2).\end{aligned}$$

The lowpass mask $\mathbf{P}(\omega)$ supported on $[-1, 1]^2$ and the multiscaling function Φ corresponding to this lowpass filter is the continuous linear box-spline B_{111} associated with the direction sets for box-splines

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

The lowpass mask $\tilde{\mathbf{P}}(\omega)$ supported on $[-2, 2]^2$ and the multiscaling function $\tilde{\Phi}$ corresponding to this lowpass filter is the continuous C^2 box-spline B_{222} associated with the direction sets for box-splines

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

The highpass analysis matrix mask $\mathbf{Q}^{(\ell)}$ and the highpass synthesis matrix mask $\tilde{\mathbf{Q}}^{(\ell)}$ will be as follows:

$$\mathbf{Q}_1^{(1)}(\omega) = \begin{bmatrix} Q_{11}^{(1)}(\omega) & Q_{12}^{(1)}(\omega) \\ Q_{21}^{(1)}(\omega) & Q_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}Q_{11}^{(1)}(\omega) = & (-0.16667x^2y^2 - 0.16667x^2y - 0.16667xy^2 + 4xy - 0.16667x - 0.16667y \\ & -0.16667)/(xy), \\ Q_{12}^{(1)}(\omega) = & 0, \\ Q_{21}^{(1)}(\omega) = & 0, \\ Q_{22}^{(1)}(\omega) = & (6.08095238x^2y^2 + 6.08095238x^2y + 6.08095238xy^2 \\ & +4xy + 6.08095238x + 6.08095238y + 6.08095238)/(xy).\end{aligned}$$

$$\mathbf{Q}_2^{(2)}(\omega) = \begin{bmatrix} Q_{11}^{(2)}(\omega) & Q_{12}^{(2)}(\omega) \\ Q_{21}^{(2)}(\omega) & Q_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}
Q_{11}^{(2)}(\omega) = & (0.1458939998842x^4y^4 + 0.1458939998842x^4y^3 + 0.1458939998842x^3y^4 \\
& + 0.64111991246710x^4y^2 - 0.8753464922584x^3y^3 + 0.64111991246710x^2y^4 \\
& + 0.64111991246710x^4y + 0.7870139127702x^3y^2 + 0.7870139127702x^2y^3 \\
& + 0.64111991246710xy^4 - 4.3467092649529x^3y + 1.291788264807x^2y^2 \\
& - 4.3467092649529xy^3 + 0.64111991246710x^3 + 0.7870139127702x^2y \\
& + 0.7870139127702xy^2 + 0.64111991246710y^3 + 0.64111991246710x^2 \\
& - 0.8753464922584xy + 0.64111991246710y^2 + 0.1458939998842x \\
& + 0.1458939998842y + 0.1458939998842)/(x^3y^3),
\end{aligned}$$

$$\begin{aligned}
Q_{12}^{(2)}(\omega) = & -(0.386501220885x^4y^4 + 0.386501220885x^4y^3 + 0.386501220885x^3y^4 \\
& + 0.5516176082468x^4y^2 + 0.0635593237181x^3y^3 + 0.5516176082468x^2y^4 \\
& + 0.5516176082468x^4y + 0.9381188299704x^3y^2 + 0.9381188299704x^2y^3 \\
& + 0.5516176082468xy^4 - 5.6922722978775x^3y + 0.7730024417716x^2y^2 \\
& - 5.6922722978775xy^3 + 0.5516176082468x^3 + 0.9381188299704x^2y \\
& + 0.9381188299704xy^2 + 0.5516176082468y^3 + 0.5516176082468x^2 \\
& + 0.06355932371819xy + 0.5516176082468y^2 + 0.386501220885x \\
& + 0.386501220885y + 0.386501220885)/(x^3y^3),
\end{aligned}$$

$$\begin{aligned}
Q_{21}^{(2)}(\omega) = & (0.11943798340545x^4y^4 + 0.11943798340545x^4y^3 + 0.11943798340545x^3y^4 \\
& + 0.22697108353401x^4y^2 - 0.7052485356495x^3y^3 + 0.22697108353401x^2y^4 \\
& + 0.22697108353401x^4y + 0.3464090670232x^3y^2 + 0.3464090670232x^2y^3 \\
& + 0.22697108353401xy^4 - 1.37316442731901x^3y + 0.2388759668109x^2y^2 \\
& - 1.37316442731901xy^3 + 0.22697108353401x^3 + 0.3464090670232x^2y \\
& + 0.3464090670232xy^2 + 0.22697108353401y^3 + 0.22697108353401x^2 \\
& - 0.7052485356495xy + 0.22697108353401y^2 + 0.11943798340545x \\
& + 0.11943798340545y + 0.11943798340545)/(x^3y^3),
\end{aligned}$$

$$\begin{aligned}
Q_{22}^{(2)}(\omega) = & -(4.92125793880315x^4y^4 + 4.92125793880315x^4y^3 + 4.92125793880315x^3y^4 \\
& - 0.41135871747178x^4y^2 + 0.93155436990932x^3y^3 - 0.41135871747178x^2y^4 \\
& - 0.41135871747178x^4y + 0.016755555556x^3y^2 + 0.016755555556x^2y^3 \\
& - 0.41135871747178xy^4 - 0.20002497299306x^3y + 8.84251587587883x^2y^2 \\
& - 0.20002497299306xy^3 - 0.41135871747178x^3 + 0.016755555556x^2y \\
& + 0.016755555556xy^2 - 0.41135871747178y^3 - 0.41135871747178x^2 \\
& + 0.93155436990932xy - 0.41135871747178y^2 + 4.92125793880315x \\
& + 4.92125793880315y + 4.92125793880315)/(x^3y^3).
\end{aligned}$$

$$\mathbf{Q}_2^{(3)}(\omega) = \begin{bmatrix} Q_{11}^{(3)}(\omega) & Q_{12}^{(3)}(\omega) \\ Q_{21}^{(3)}(\omega) & Q_{22}^{(3)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned} Q_{11}^{(3)}(\omega) = & (0.64111974410947x^4y^6 + 0.64111974410947x^4y^5 + 0.64111974410947x^3y^6 \\ & + 0.14589396157254x^4y^4 - 4.34670812350973x^3y^5 + 0.14589396157254x^4y^3 \\ & + 0.7870137061009x^3y^4 + 0.64111974410947x^2y^5 - 0.87534626239298x^3y^3 \\ & + 0.7870137061009x^2y^4 + 0.14589396157254x^3y^2 + 1.29178792558427x^2y^3 \\ & + 0.14589396157254xy^4 + 0.7870137061009x^2y^2 - 0.87534626239298xy^3 \\ & + 0.64111974410947x^2y + 0.7870137061009xy^2 + 0.14589396157254y^3 \\ & - 4.34670812350973xy + 0.14589396157254y^2 + 0.64111974410947x \\ & + 0.64111974410947y + 0.64111974410947)/(xy^3), \end{aligned}$$

$$\begin{aligned} Q_{12}^{(3)}(\omega) = & -(0.55161760824688x^4y^6 + 0.55161760824688x^4y^5 + 0.55161760824688x^3y^6 \\ & + 0.3865012208858x^4y^4 - 5.69227229787754x^3y^5 + 0.3865012208858x^4y^3 \\ & + 0.93811882997045x^3y^4 + 0.55161760824688x^2y^5 + 0.7586656476x^3y^3 \\ & + 0.93811882997045x^2y^4 + 0.3865012208858x^3y^2 + 0.7730024417716x^2y^3 \\ & + 0.3865012208858xy^4 + 0.93811882997045x^2y^2 + 0.7586656476xy^3 \\ & + 0.55161760824688x^2y + 0.93811882997045xy^2 + 0.3865012208858y^3 \\ & - 5.69227229787754xy + 0.3865012208858y^2 + 0.55161760824688x \\ & + 0.55161760824688y + 0.55161760824688)/(xy^3), \end{aligned}$$

$$\begin{aligned} Q_{21}^{(3)}(\omega) = & (0.22697108353401x^4y^6 + 0.22697108353401x^4y^5 + 0.22697108353401x^3y^6 \\ & + 0.11943798340545x^4y^4 - 1.37316442731901x^3y^5 + 0.11943798340545x^4y^3 \\ & + 0.34640906702324x^3y^4 + 0.22697108353401x^2y^5 - 0.70524853564952x^3y^3 \\ & + 0.34640906702324x^2y^4 + 0.11943798340545x^3y^2 + 0.2388759668109x^2y^3 \\ & + 0.11943798340545xy^4 + 0.34640906702324x^2y^2 - 0.70524853564952xy^3 \\ & + 0.22697108353401x^2y + 0.34640906702324xy^2 + 0.11943798340545y^3 \\ & - 1.37316442731901xy + 0.11943798340545y^2 + 0.22697108353401x \\ & + 0.22697108353401y + 0.22697108353401)/(xy^3), \end{aligned}$$

$$\begin{aligned}
Q_{22}^{(3)}(\omega) = & (0.4113587184538x^4y^6 + 0.4113587184538x^4y^5 + 0.4113587184538x^3y^6 \\
& - 4.92125795055151x^4y^4 + 0.20002497347057x^3y^5 - 4.92125795055151x^4y^3 \\
& - 4.509899223384825x^3y^4 + 0.4113587184538x^2y^5 - 0.93155437213319x^3y^3 \\
& - 4.509899223384825x^2y^4 - 4.92125795055151x^3y^2 - 8.84251589698828x^2y^3 \\
& - 4.92125795055151xy^4 - 4.509899223384825x^2y^2 - 0.93155437213319xy^3 \\
& + 0.4113587184538x^2y - 4.509899223384825xy^2 - 4.92125795055151y^3 \\
& + 0.20002497347057xy - 4.92125795055151y^2 + 0.4113587184538x \\
& + 0.4113587184538y + 0.4113587184538)/(xy^3).
\end{aligned}$$

$$\mathbf{Q}_2^{(4)}(\omega) = \begin{bmatrix} Q_{11}^{(4)}(\omega) & Q_{12}^{(4)}(\omega) \\ Q_{21}^{(4)}(\omega) & Q_{22}^{(4)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}
Q_{11}^{(4)}(\omega) = & (0.64111974410947x^6y^4 + 0.64111974410947x^6y^3 + 0.64111974410947x^5y^4 \\
& - 4.34670812350973x^5y^3 + 0.14589396157254x^4y^4 + 0.64111974410947x^5y^2 \\
& + 0.7870137061009x^4y^3 - 0.14589396157254x^3y^4 + 0.7870137061009x^4y^2 \\
& - 0.87534626239298x^3y^3 + 0.14589396157254x^4y + 1.29178792558427x^3y^2 \\
& + 0.14589396157254x^2y^3 - 0.87534626239298x^3y + 0.7870137061009x^2y^2 \\
& + 0.14589396157254x^3 + 0.7870137061009x^2y + 0.64111974410947xy^2 \\
& + 0.14589396157254x^2 - 4.34670812350973xy + 0.64111974410947x \\
& + 0.64111974410947y + 0.64111974410947)/(x^3y),
\end{aligned}$$

$$\begin{aligned}
Q_{12}^{(4)}(\omega) = & -(0.55161760824688x^6y^4 + 0.55161760824688x^6y^3 + 0.55161760824688x^5y^4 \\
& - 5.69227229787754x^5y^3 + 0.3865012208858x^4y^4 + 0.55161760824688x^5y^2 \\
& + 0.93811882997045x^4y^3 + 0.3865012208858x^3y^4 + 0.93811882997045x^4y^2 \\
& + 0.06355932371819x^3y^3 + 0.3865012208858x^4y + 9.226819352x^3y^2 \\
& + 0.3865012208858x^2y^3 + 0.06355932371819x^3y + 0.93811882997045x^2y^2 \\
& + 0.3865012208858x^3 + 0.93811882997045x^2y + 0.55161760824688xy^2 \\
& + 0.3865012208858x^2 - 5.69227229787754xy + 0.55161760824688x \\
& + 0.55161760824688y + 0.55161760824688)/(x^3y),
\end{aligned}$$

$$\begin{aligned}
Q_{21}^{(4)}(\omega) = & (0.226971083x^6y^4 + 0.226971083x^6y^3 + 0.226971083x^5y^4 \\
& - 0.535386627x^5y^3 + 0.119437983x^4y^4 + 0.226971083x^5y^2 + 0.346409067x^4y^3 \\
& + 0.119437983x^3y^4 + 0.346409067x^4y^2 - 0.7052485x^3y^3 + 0.119437983x^4y \\
& + 2.851304568x^3y^2 + 0.119437983x^2y^3 - 0.7052485x^3y + 0.346409067x^2y^2 \\
& + 0.119437983x^3 + 0.346409067x^2y + 0.226971083xy^2 + 0.119437983x^2 \\
& - 0.535386627xy + 0.226971083x + 0.226971083y + 0.226971083)/(x^3y),
\end{aligned}$$

$$\begin{aligned}
Q_{22}^{(4)}(\omega) = & (0.41135872827404x^6y^4 + 0.41135872827404x^6y^3 + 0.41135872827404x^5y^4 \\
& + 0.2000249782457x^5y^3 - 4.92125806803512x^4y^4 + 0.41135872827404x^5y^2 \\
& - 4.50989933942597x^4y^3 - 4.92125806803512x^3y^4 - 4.50989933942597x^4y^2 \\
& - 0.93155439437189x^3y^3 - 4.92125806803512x^4y - 8.84251610808283x^3y^2 \\
& - 4.92125806803512x^2y^3 - 0.93155439437189x^3y - 4.50989933942597x^2y^2 \\
& - 4.92125806803512x^3 - 4.50989933942597x^2y + 0.41135872827404xy^2 \\
& - 4.92125806803512x^2 + 0.2000249782457xy + 0.41135872827404x \\
& + 0.41135872827404y + 0.41135872827404)/(x^3y).
\end{aligned}$$

and

$$\tilde{\mathbf{Q}}_1^{(1)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(1)}(\omega) & \tilde{Q}_{12}^{(1)}(\omega) \\ \tilde{Q}_{21}^{(1)}(\omega) & \tilde{Q}_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}
\tilde{Q}_{11}^{(1)}(\omega) = & (9.381043145x^4y^4 + 0.129326923075x^4y^3 + 0.129326923075x^3y^4 \\
& + 9.381043145x^4y^2 + 0.2188365651x^3y^3 + 9.381043145x^2y^4 + 0.2188365651x^3y^2 \\
& + 0.2188365651x^2y^3 + 0.129326923075x^3y + 13.9668227425x^2y^2 + 0.129326923075xy^3 \\
& + 0.2188365651x^2y + 0.2188365651xy^2 + 9.381043145x^2 + 0.2188365651xy \\
& + 9.381043145y^2 + 0.129326923075x + 0.129326923075y + 9.381043145)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{12}^{(1)}(\omega) = & (-0.6156278895x^4y^4 + 0.321381579x^4y^3 + 0.321381579x^3y^4 \\
& - 0.6156278895x^4y^2 + 0.19660194175x^3y^3 - 0.6156278895x^2y^4 + 0.19660194175x^3y^2 \\
& + 0.19660194175x^2y^3 + 0.321381579x^3y - 0.7075189115x^2y^2 + 0.321381579xy^3 \\
& + 0.19660194175x^2y + 0.19660194175xy^2 - 0.6156278895x^2 + 0.19660194175xy \\
& - 0.6156278895y^2 + 0.321381579x + 0.321381579y - 0.6156278895)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{21}^{(1)}(\omega) = & (1.8897507525x^4y^4 + 0.0105263157875x^4y^3 + 0.0105263157875x^3y^4 \\
& + 1.8897507525x^4y^2 + 0.0158898305075x^3y^3 + 1.8897507525x^2y^4 + 0.0158898305075x^3y^2 \\
& + 0.0158898305075x^2y^3 + 0.0105263157875x^3y + 12.90277912x^2y^2 + 0.0105263157875xy^3 \\
& + 0.0158898305075x^2y + 0.0158898305075xy^2 + 1.8897507525x^2 + 0.0158898305075xy \\
& + 1.8897507525y^2 + 0.0105263157875x + 0.0105263157875y + 1.8897507525)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{22}^{(1)}(\omega) = & (-0.123869110175x^4y^4 - 0.0164670658675x^4y^3 - 0.0164670658675x^3y^4 \\ & -0.123869110175x^4y^2 + 0.2019119351x^3y^3 - 0.123869110175x^2y^4 + 0.2019119351x^3y^2 \\ & +0.2019119351x^2y^3 - 0.0164670658675x^3y - 0.8709272185x^2y^2 - 0.0164670658675xy^3 \\ & +0.2019119351x^2y + 0.2019119351xy^2 - 0.123869110175x^2 + 0.2019119351xy \\ & -0.123869110175y^2 - 0.0164670658675x - 0.0164670658675y - 0.123869110175)/(x^2y^2).\end{aligned}$$

$$\tilde{\mathbf{Q}}_2^{(2)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(2)}(\omega) & \tilde{Q}_{12}^{(2)}(\omega) \\ \tilde{Q}_{21}^{(2)}(\omega) & \tilde{Q}_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$\tilde{Q}_{11}^{(2)}(\omega) = (0.06545830143113x^2y^2 + xy + 0.06545830143113)/(x^2y^2),$$

$$\tilde{Q}_{12}^{(2)}(\omega) = (0.004200338475(x^2y^2 + 1))/(x^2y^2),$$

$$\tilde{Q}_{21}^{(2)}(\omega) = (2.751333038(x^2y^2 + 1))/(x^2y^2),$$

$$\tilde{Q}_{22}^{(2)}(\omega) = (-0.185750627075x^2y^2 + xy - 0.185750627075)/(x^2y^2).$$

$$\tilde{\mathbf{Q}}_2^{(3)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(3)}(\omega) & \tilde{Q}_{12}^{(3)}(\omega) \\ \tilde{Q}_{21}^{(3)}(\omega) & \tilde{Q}_{22}^{(3)}(\omega) \end{bmatrix},$$

where

$$\tilde{Q}_{11}^{(3)}(\omega) = 0.0751593245x^2 + 0.07515932448 + 0.25x,$$

$$\tilde{Q}_{12}^{(3)}(\omega) = 0.004200338475x^2 + 0.004200338475,$$

$$\tilde{Q}_{21}^{(3)}(\omega) = 2.751333038x^2 + 2.751333038,$$

$$\tilde{Q}_{22}^{(3)}(\omega) = -0.185750627 - 0.185750627x^2 + 0.25x.$$

$$\tilde{\mathbf{Q}}_2^{(4)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(4)}(\omega) & \tilde{Q}_{12}^{(4)}(\omega) \\ \tilde{Q}_{21}^{(4)}(\omega) & \tilde{Q}_{22}^{(4)}(\omega) \end{bmatrix},$$

where

$$\tilde{Q}_{11}^{(4)}(\omega) = 0.0751593245y^2 + 0.7515932448 + 0.25y,$$

$$\tilde{Q}_{12}^{(4)}(\omega) = 0.004200338475y^2 + 0.004200338475,$$

$$\tilde{Q}_{21}^{(4)}(\omega) = 2.751333038y^2 + 2.751333038,$$

$$\tilde{Q}_{22}^{(4)}(\omega) = -0.185750627 - 0.185750627y^2 + 0.25y.$$

In the following subsection we consider a 3-step algorithm.

6.2.2 3-Step Multiple Bi-frame Multiresolution Algorithm

In this subsection, we consider 3-step frame algorithm. For given 1×2 row vector $\{\mathbf{v}\}$ (or given triangular mesh C), Fig. 6.6 shows the multiresolution decomposition algorithm. Where \mathbf{B} , \mathbf{M} , \mathbf{N} , \mathbf{A} , \mathbf{C} , \mathbf{H} , \mathbf{J} , \mathbf{U} , \mathbf{W} , and \mathbf{T} are 2×2 matrices and the entries of these matrices are some constants in \mathbb{R} to be determined.

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \mathbf{N} = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}, \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \mathbf{W} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}, \mathbf{J} = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}, \mathbf{T} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}.$$

3-step Dyadic Multiple Frame Decomposition Algorithm:

$$\text{Step 1. } \begin{cases} \mathbf{v}'' = \{\mathbf{v} - (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) \mathbf{M}\} \mathbf{B}^{-1}, \\ \mathbf{g}'' = \mathbf{v} - (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) \mathbf{N}; \end{cases} \quad (6.26)$$

$$\text{Step 2. } \tilde{\mathbf{e}} = \mathbf{e} - (\mathbf{v}_0'' + \mathbf{v}_1'') \mathbf{A} - (\mathbf{v}_2'' + \mathbf{v}_3'') \mathbf{C} - (\mathbf{g}_0'' + \mathbf{g}_1'') \mathbf{H} - (\mathbf{g}_2'' + \mathbf{g}_3'') \mathbf{J}; \quad (6.27)$$

$$\text{Step 3. } \begin{cases} \tilde{\mathbf{v}} = \mathbf{v}'' - (\tilde{\mathbf{e}}_0 + \tilde{\mathbf{e}}_1 + \tilde{\mathbf{e}}_2 + \tilde{\mathbf{e}}_3 + \tilde{\mathbf{e}}_4 + \tilde{\mathbf{e}}_5) \mathbf{W}, \\ \tilde{\mathbf{g}} = \mathbf{g}'' - (\tilde{\mathbf{e}}_0 + \tilde{\mathbf{e}}_1 + \tilde{\mathbf{e}}_2 + \tilde{\mathbf{e}}_3 + \tilde{\mathbf{e}}_4 + \tilde{\mathbf{e}}_5) \mathbf{U}. \end{cases} \quad (6.28)$$

We replace all \mathbf{v} , which is associated with type V nodes of \mathcal{M}_0 by \mathbf{v}'' and \mathbf{g}'' . Then, in Step 2, we use the obtained \mathbf{v}'' and \mathbf{g}'' to obtain other three highpass outputs $\{\tilde{\mathbf{e}}\} (= \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(1)}\} \cup \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(2)}\} \cup \{\tilde{\mathbf{e}}_{\mathbf{k}}^{(3)}\})$ that are associated with type E nodes of \mathcal{M}_0 , by $\tilde{\mathbf{e}}$. After that in Step 3, we use the obtained $\tilde{\mathbf{e}}$ in Step 2 to update all \mathbf{v}'' and \mathbf{g}'' in Step 1 by $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{g}}$.

The reconstruction algorithm is the backward algorithm of the decomposition algorithm as

shown in Fig. 6.7. We give the multiresolution reconstruction algorithm, where \mathbf{B} , \mathbf{M} , \mathbf{N} , \mathbf{A} , \mathbf{C} , \mathbf{H} , \mathbf{J} , and \mathbf{T} are the same 2×2 matrices in the decomposition algorithm.

3-step Dyadic Multiple Frame Reconstruction Algorithm:

$$\text{Step 1. } \begin{cases} \mathbf{v}'' = \tilde{\mathbf{v}} + (\tilde{\mathbf{e}}_0 + \tilde{\mathbf{e}}_1 + \tilde{\mathbf{e}}_2 + \tilde{\mathbf{e}}_3 + \tilde{\mathbf{e}}_4 + \tilde{\mathbf{e}}_5)\mathbf{W}, \\ \mathbf{g}'' = \tilde{\mathbf{g}} + (\tilde{\mathbf{e}}_0 + \tilde{\mathbf{e}}_1 + \tilde{\mathbf{e}}_2 + \tilde{\mathbf{e}}_3 + \tilde{\mathbf{e}}_4 + \tilde{\mathbf{e}}_5)\mathbf{U}; \end{cases} \quad (6.29)$$

$$\text{Step 2. } \mathbf{e} = \tilde{\mathbf{e}} + (\mathbf{v}''_0 + \mathbf{v}''_1)\mathbf{A} + (\mathbf{v}''_2 + \mathbf{v}''_3)\mathbf{C} + (\mathbf{g}''_0 + \mathbf{g}''_1)\mathbf{H} + (\mathbf{g}''_2 + \mathbf{g}''_3)\mathbf{J}; \quad (6.30)$$

$$\text{Step 3. } \mathbf{v} = \{\mathbf{v}''\mathbf{B} + (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5)\mathbf{M}\}\mathbf{T} \\ + \{\mathbf{g}'' + (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5)\mathbf{N}\}(\mathbf{I}_2 - \mathbf{T}). \quad (6.31)$$

First in Step 1, we replace $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{g}}$ by \mathbf{v}'' and \mathbf{g}'' . Then in Step 2 with \mathbf{v}'' and \mathbf{g}'' obtained

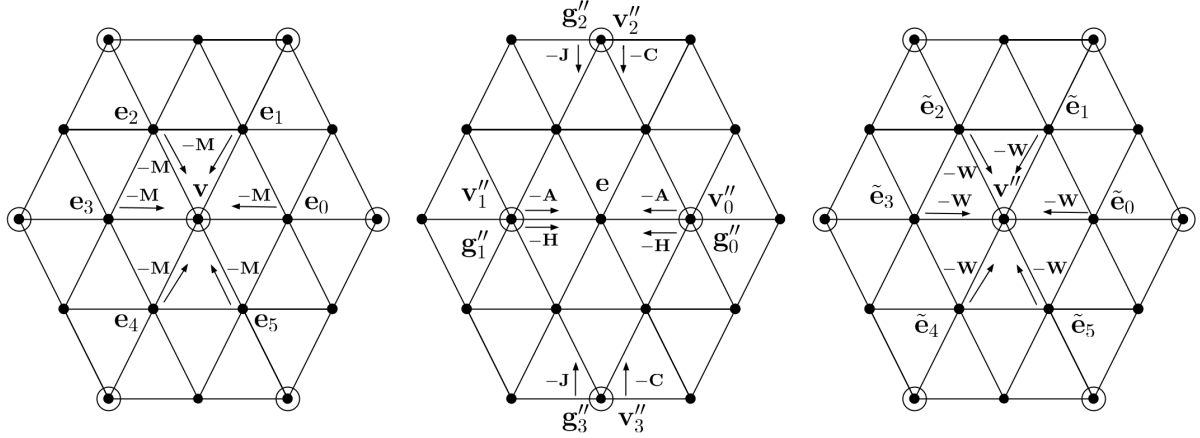


Figure 6.6: Left: template to obtain \mathbf{v}'' in Decomposition Algorithm Step 1 (template to obtain \mathbf{g}'' is similar with $-\mathbf{M}$ replaced by $-\mathbf{N}$); Middle: Decomposition Algorithm Step 2; Right: In Decomposition Algorithm Step 3, template to get lowpass output $\tilde{\mathbf{v}}$ (template to obtain the highpass output $\tilde{\mathbf{g}}$ is like \mathbf{v}'' and $-\mathbf{W}$ replaced by \mathbf{g}'' and $-\mathbf{U}$, receptively).

in Step 1, we replace $\tilde{\mathbf{e}}$ by \mathbf{e} . After that in Step 3, we use the obtained \mathbf{e} in Step 2 to replace all \mathbf{v}'' , \mathbf{g}'' in Step 1 by \mathbf{v} . So, the inputs here are $\tilde{\mathbf{v}}$, $\tilde{\mathbf{g}}$, $\tilde{\mathbf{e}}$ and the outputs are \mathbf{v}'' , \mathbf{g}'' , \mathbf{v} , \mathbf{e} . As before, \mathbf{B} , \mathbf{M} , \mathbf{N} , \mathbf{A} , \mathbf{C} , \mathbf{H} , \mathbf{J} , \mathbf{W} , \mathbf{U} , and \mathbf{T} are 2×2 matrices and the entries of these matrices are some constants in \mathbb{R} .

Similar to the previous (2-Step) algorithm, we can follow the detailed calculations in [73] which used to obtain 1-D multifilters associated with some given templates to get the multifilter banks $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$ and $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}, \tilde{\mathbf{Q}}^{(4)}\}$ corresponding to this (3-step) algorithm. $\mathbf{G}_0(\omega)$, $\tilde{\mathbf{G}}_0(\omega)$, $\mathbf{G}_1(\omega)$, and $\tilde{\mathbf{G}}_1(\omega)$ are the same matrices which are used in (2-step) algorithm. $\omega = (\omega_1, \omega_2)$, we use x and y to denote $e^{-i\omega_1}$, $e^{-i\omega_2}$

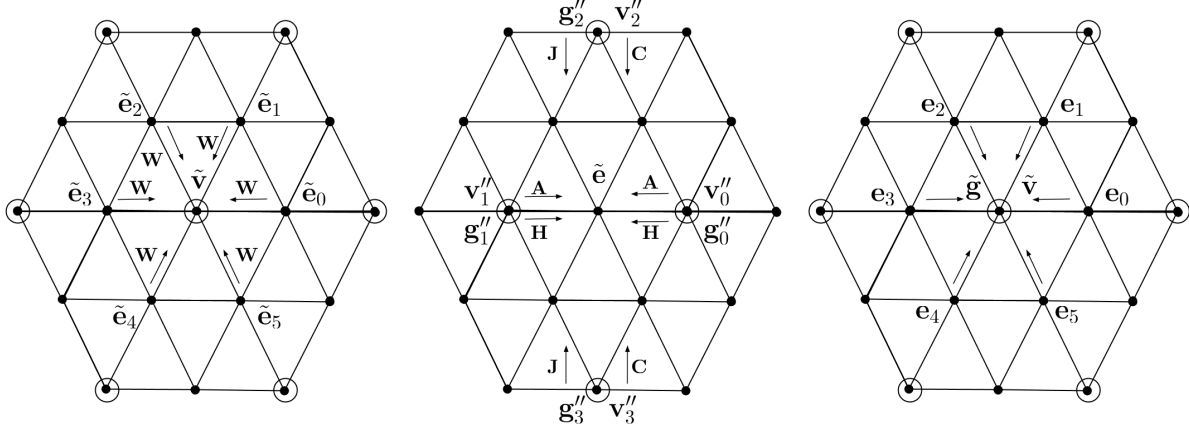


Figure 6.7: Left: Reconstruction Algorithm Step 1; Middle: Reconstruction Algorithm Step 2; Right: Reconstruction Step 3.

respectively:

$$x = e^{-i\omega_1}, \quad y = e^{-i\omega_2}.$$

We derive the analysis lowpass multifilter \mathbf{P}_{k_1, k_2} which is satisfy the sum rule of order 1, namely:

$$\begin{aligned} \mathbf{P}_{0,0} &= 4 (\mathbf{B}^{-1})^T + 24 \mathbf{W}^T \mathbf{H}^T + 24 \mathbf{W}^T \mathbf{A}^T (\mathbf{B}^{-1})^T, \\ \mathbf{P}_{1,0} = \mathbf{P}_{0,1} = \mathbf{P}_{0,-1} = \mathbf{P}_{-1,0} = \mathbf{P}_{-1,-1} = \mathbf{P}_{1,1} &= -4 (\mathbf{B}^{-1})^T \mathbf{M}^T - 4 \mathbf{W}^T \\ &\quad - 28 \mathbf{W}^T \mathbf{A}^T (\mathbf{B}^{-1})^T \mathbf{M}^T - 8 \mathbf{W}^T \mathbf{C}^T (\mathbf{B}^{-1})^T \mathbf{M}^T - 28 \mathbf{W}^T \mathbf{H}^T \mathbf{N}^T - 8 \mathbf{W}^T \mathbf{J}^T \mathbf{N}^T, \\ \mathbf{P}_{2,0} = \mathbf{P}_{0,2} = \mathbf{P}_{0,-2} = \mathbf{P}_{-2,0} = \mathbf{P}_{-2,-2} = \mathbf{P}_{2,2} &= 8 \mathbf{W}^T \mathbf{C}^T (\mathbf{B}^{-1})^T + 4 \mathbf{W}^T \mathbf{A}^T (\mathbf{B}^{-1})^T \\ &\quad + 8 \mathbf{W}^T \mathbf{J}^T + 4 \mathbf{W}^T \mathbf{H}^T, \\ \mathbf{P}_{1,2} = \mathbf{P}_{2,1} = \mathbf{P}_{-1,1} = \mathbf{P}_{1,-1} = \mathbf{P}_{-1,-2} = \mathbf{P}_{-2,-1} &= -8 \mathbf{W}^T \mathbf{A}^T (\mathbf{B}^{-1})^T \mathbf{M}^T \\ &\quad - 16 \mathbf{W}^T \mathbf{C}^T (\mathbf{B}^{-1})^T \mathbf{M}^T - 8 \mathbf{W}^T \mathbf{H}^T \mathbf{N}^T - 16 \mathbf{W}^T \mathbf{J}^T \mathbf{N}^T, \\ \mathbf{P}_{3,0} = \mathbf{P}_{0,3} = \mathbf{P}_{-3,-3} = \mathbf{P}_{3,3} = \mathbf{P}_{0,-3} = \mathbf{P}_{-3,0} = \mathbf{P}_{1,3} = \mathbf{P}_{3,1} = \mathbf{P}_{2,3} = \mathbf{P}_{3,2} = \mathbf{P}_{-1,-3} \\ &= \mathbf{P}_{-3,-1} = \mathbf{P}_{-2,-3} = \mathbf{P}_{-3,-2} = \mathbf{P}_{-2,1} = \mathbf{P}_{2,-1} = \mathbf{P}_{-1,2} = \mathbf{P}_{1,-2} \\ &= -8 \mathbf{W}^T \mathbf{C}^T (\mathbf{B}^{-1})^T \mathbf{M}^T - 4 \mathbf{W}^T \mathbf{A}^T (\mathbf{B}^{-1})^T \mathbf{M}^T - 8 \mathbf{W}^T \mathbf{J}^T \mathbf{N}^T - 4 \mathbf{W}^T \mathbf{H}^T \mathbf{N}^T. \end{aligned}$$

and we derive the synthesis lowpass multifilter $\tilde{\mathbf{P}}_{k_1, k_2}$ which is satisfy the sum rule of order 1, namely:

$$\begin{aligned} \tilde{\mathbf{P}}_{0,0} &= \mathbf{B}\mathbf{T} + 6 (\mathbf{A}\mathbf{M}\mathbf{T} + \mathbf{A}\mathbf{N}(\mathbf{I}_2 - \mathbf{T})), \\ \tilde{\mathbf{P}}_{1,0} = \tilde{\mathbf{P}}_{0,1} = \tilde{\mathbf{P}}_{0,-1} = \tilde{\mathbf{P}}_{-1,0} = \tilde{\mathbf{P}}_{-1,-1} = \tilde{\mathbf{P}}_{1,1} &= \mathbf{A}, \\ \tilde{\mathbf{P}}_{2,0} = \tilde{\mathbf{P}}_{0,2} = \tilde{\mathbf{P}}_{0,-2} = \tilde{\mathbf{P}}_{-2,0} = \tilde{\mathbf{P}}_{-2,-2} = \tilde{\mathbf{P}}_{2,2} &= \mathbf{A}\mathbf{M}\mathbf{T} \\ &\quad + \mathbf{A}\mathbf{N}(\mathbf{I}_2 - \mathbf{T}) + 2 (\mathbf{C}\mathbf{M}\mathbf{T} + \mathbf{C}\mathbf{N}(\mathbf{I}_2 - \mathbf{T})), \\ \tilde{\mathbf{P}}_{1,2} = \tilde{\mathbf{P}}_{2,1} = \tilde{\mathbf{P}}_{-1,1} = \tilde{\mathbf{P}}_{1,-1} = \tilde{\mathbf{P}}_{-1,-2} = \tilde{\mathbf{P}}_{-2,-1} &= \mathbf{C}. \end{aligned}$$

We denote

$$\left\{ \begin{array}{l} \mathbf{G}_2(\boldsymbol{\omega}) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & -(1+xy)\mathbf{W}^T & -(1+\frac{1}{x})\mathbf{W}^T & -(1+\frac{1}{y})\mathbf{W}^T \\ \mathbf{0}_2 & \mathbf{I}_2 & -(1+xy)\mathbf{U}^T & -(1+\frac{1}{x})\mathbf{U}^T & -(1+\frac{1}{y})\mathbf{U}^T \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \\ \\ \mathbf{G}_1(\boldsymbol{\omega}) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -(1+\frac{1}{xy})\mathbf{A}^T - (\frac{1}{x} + \frac{1}{y})\mathbf{C}^T & -(1+\frac{1}{xy})\mathbf{H}^T - (\frac{1}{x} + \frac{1}{y})\mathbf{J}^T & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -(1+x)\mathbf{A}^T - (xy + \frac{1}{y})\mathbf{C}^T & -(1+x)\mathbf{H}^T - (xy + \frac{1}{y})\mathbf{J}^T & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ -(1+y)\mathbf{A}^T - (xy + \frac{1}{x})\mathbf{C}^T & -(1+y)\mathbf{H}^T - (xy + \frac{1}{x})\mathbf{J}^T & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \\ \\ \mathbf{G}_0(\boldsymbol{\omega}) = \begin{bmatrix} (\mathbf{B}^{-1})^T & -(1+xy)(\mathbf{B}^{-1})^T\mathbf{M}^T & -(1+\frac{1}{x})(\mathbf{B}^{-1})^T\mathbf{M}^T & -(1+\frac{1}{y})(\mathbf{B}^{-1})^T\mathbf{M}^T \\ \mathbf{I}_2 & -(1+xy)\mathbf{N}^T & -(1+\frac{1}{x})\mathbf{N}^T & -(1+\frac{1}{y})\mathbf{N}^T \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}. \end{array} \right. \quad (6.32)$$

and

$$\left\{ \begin{array}{l} \tilde{\mathbf{G}}_2(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1 + \frac{1}{xy})\mathbf{W} & (1 + \frac{1}{xy})\mathbf{U} & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1 + x)\mathbf{W} & (1 + x)\mathbf{U} & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ (1 + y)\mathbf{W} & (1 + y)\mathbf{U} & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \\ \\ \tilde{\mathbf{G}}_1(\omega) = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & (1 + xy)\mathbf{A} + (x + y)\mathbf{C} & (1 + \frac{1}{x})\mathbf{A} + (\frac{1}{xy} + y)\mathbf{C} & (1 + \frac{1}{y})\mathbf{A} + (\frac{1}{xy} + x)\mathbf{C} \\ \mathbf{0}_2 & \mathbf{I}_2 & (1 + xy)\mathbf{H} + (x + y)\mathbf{J} & (1 + \frac{1}{x})\mathbf{H} + (\frac{1}{xy} + y)\mathbf{J} & (1 + \frac{1}{y})\mathbf{H} + (\frac{1}{xy} + x)\mathbf{J} \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}, \\ \\ \tilde{\mathbf{G}}_0(\omega) = \begin{bmatrix} \mathbf{BT} & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (\mathbf{I}_2 - \mathbf{T}) & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1 + \frac{1}{xy})(\mathbf{MT} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T})) & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ (1 + x)(\mathbf{MT} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T})) & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ (1 + y)(\mathbf{MT} + \mathbf{N}(\mathbf{I}_2 - \mathbf{T})) & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}. \end{array} \right. \quad (6.33)$$

The analysis multifilter bank $\{\mathbf{P}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(4)}\}$ and the synthesis multifilter bank $\{\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(2)}, \tilde{\mathbf{Q}}^{(3)}, \tilde{\mathbf{Q}}^{(4)}\}$ can be written as the following:

$$[\mathbf{P}(\omega), \mathbf{Q}^{(1)}(\omega), \mathbf{Q}^{(2)}(\omega), \mathbf{Q}^{(3)}(\omega), \mathbf{Q}^{(4)}(\omega)]^T = \mathbf{G}_2(2\omega)\mathbf{G}_1(2\omega)\mathbf{G}_0(2\omega)\mathbf{I}_0(\omega), \quad (6.34)$$

$$[\tilde{\mathbf{P}}(\omega), \tilde{\mathbf{Q}}^{(1)}(\omega), \tilde{\mathbf{Q}}^{(2)}(\omega), \tilde{\mathbf{Q}}^{(3)}(\omega), \tilde{\mathbf{Q}}^{(4)}(\omega)]^T = \frac{1}{4}\tilde{\mathbf{G}}_2(2\omega)\tilde{\mathbf{G}}_1(2\omega)\tilde{\mathbf{G}}_0(2\omega)\mathbf{I}_0(\omega), \quad (6.35)$$

Next step, we solve the system of equations for a sum rule of order one for \mathbf{P} and $\tilde{\mathbf{P}}$ and a vanishing moment of order one for the analysis multifilter highpass bank to obtain the parameters.

We have 27 free parameters which are

$$b_{22}, c_{12}, c_{21}, h_{11}, h_{21}, j_{11}, j_{21}, m_{12}, n_{22}, t_{11}, t_{12}, t_{21}, t_{22}, a_{22}, c_{22}, h_{12}, h_{22}, j_{12}, j_{22}, w_{11}, w_{21}, w_{12}, w_{22}, \\ u_{11}, u_{21}, u_{12}, u_{22}.$$

and the other parameters are

$$a_{11} = 0, a_{12} = -c_{12}, a_{21} = 0, b_{11} = 4, b_{12} = -6m_{12}, b_{21} = 0, c_{11} = \frac{1}{2}, m_{11} = \frac{-1}{2}, m_{21} = 0, \\ m_{22} = -\frac{1}{6}(6b_{22}h_{21}n_{22} + 6b_{22}j_{21}n_{22} - b_{22}h_{21} - b_{22}j_{21} - c_{21}), n_{11} = \frac{1}{6}, n_{12} = 0, n_{21} = 0.$$

If we choose

$$\begin{aligned} & \left[b_{22}, c_{12}, c_{21}, h_{11}, h_{21}, j_{11}, j_{21}, m_{12}, n_{22}, t_{11}, t_{12}, t_{21}, t_{22}, a_{22}, c_{22}, h_{12}, h_{22}, j_{12}, j_{22}, w_{11}, w_{21}, w_{12}, \right. \\ & \left. w_{22}, u_{11}, u_{21}, u_{12}, u_{22} \right] = \left[\frac{597}{1019}, \frac{-87}{334}, \frac{443}{715}, \frac{31}{619}, \frac{-299}{553}, \frac{54}{91}, \frac{1}{333}, \frac{216}{653}, \frac{17}{523}, \frac{155}{483}, \frac{104}{441}, \frac{143}{261}, \right. \\ & \left. \frac{454}{1177}, \frac{301}{809}, \frac{-123}{400}, \frac{99}{217}, \frac{113}{247}, \frac{181}{199}, \frac{211}{280}, \frac{1}{11861}, \frac{1}{4209}, \frac{167}{1642}, \frac{-49}{321}, \frac{334}{519}, \frac{743}{1746}, \frac{47}{237}, \frac{41}{383} \right]. \end{aligned}$$

By using the smoothness formula, we can obtain $\tilde{\Phi} \in W^2$ and $\Phi \in W^{0.6758}$.

The lowpass analysis mask \mathbf{P} and the lowpass synthesis mask $\tilde{\mathbf{P}}$ will be as follows:

$$\mathbf{P}(\omega) = \begin{bmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned} P_{11}(\omega) = & (0.0007291044600y + 0.00072910446x + 0.00072910446x^6y^6 + 0.00072910446x^6y^5 \\ & + 0.00072910446x^5y^6 + 0.00072910446x^6y^4 - 0.00727505187x^5y^5 + 0.00072910446x^4y^6 \\ & + 0.00072910446x^6y^3 + 0.001458249927x^5y^4 + 0.001458249927x^4y^5 + 0.00072910446x^3y^6 \\ & - 0.00727505187x^5y^3 + 0.1290695325x^4y^4 - 0.00727505187x^3y^5 + 0.0072910446x^5y^2 \\ & + 0.1290695325x^4y^3 + 0.1290695325x^3y^4 + 0.00072910446x^2y^5 + 0.01458249927x^4y^2 \\ & + 0.2473583247x^3y^3 + 0.001458249927x^2y^4 + 0.00072910446x^4y + 0.1290695325x^3y^2 \\ & + 0.1290695325x^2y^3 + 0.00072910446xy^4 - 0.00727505187x^3y + 0.1290695325x^2y^2 \\ & - 0.00727505187xy^3 + 0.001458249927x^2y + 0.001458249927xy^2 - 0.00727505187xy \\ & + 0.00072910446y^2 + 0.00072910446y^3 + 0.00072910446x^2 + 0.00072910446x^3 \\ & + 0.00072910446)/(x^3y^3), \end{aligned}$$

$$\begin{aligned} P_{12}(\omega) = & (0.00072910446y + 0.00072910446x + 0.00072910446x^6y^6 + 0.00072910446x^6y^5 \\ & + 0.00072910446x^5y^6 + 0.00072910446x^6y^4 - 0.00727505187x^5y^5 + 0.00072910446x^4y^6 \\ & + 0.00072910446x^6y^3 + 0.001458249927x^5y^4 + 0.001458249927x^4y^5 + 0.00072910446x^3y^6 \\ & - 0.00727505187x^5y^3 + 0.1290695325x^4y^4 - 0.00727505187x^3y^5 + 0.00072910446x^5y^2 \\ & + 0.1290695325x^4y^3 + 0.1290695325x^3y^4 + 0.00072910446x^2y^5 + 0.001458249927x^4y^2 \\ & + 0.2473583247x^3y^3 + 0.001458249927x^2y^4 + 0.00072910446x^4y + 0.1290695325x^3y^2 \\ & + 0.1290695325x^2y^3 + 0.00072910446xy^4 - 0.727505187x^3y + 0.1290695325x^2y^2 \\ & - 0.00727505187xy^3 + 0.001458249927x^2y + 0.001458249927xy^2 - 0.00727505187xy \\ & + 0.00072910446y^2 + 0.00072910446y^3 + 0.00072910446x^2 + 0.00072910446x^3 \\ & + 0.00072910446)/(x^3y^3), \end{aligned}$$

$$\begin{aligned}
P_{21}(\omega) = & (-0.1342718440y - 0.1342718440x - 0.1342718440x^6y^6 - 0.1342718440x^6y^5 \\
& - 0.1342718440x^5y^6 - 0.1342718440x^6y^4 + 1.058689499x^5y^5 - 0.1342718440x^4y^6 \\
& - 0.1342718440x^6y^3 - 0.2685326350x^5y^4 - 0.2685326350x^4y^5 - 0.1342718440x^3y^6 \\
& + 1.058689499x^5y^3 - 0.5886496210x^4y^4 + 1.058689499x^3y^5 - 0.1342718440x^5y^2 \\
& - 0.5886496210x^4y^3 - 0.5886496210x^3y^4 - 0.1342718440x^2y^5 - 0.2685326350x^4y^2 \\
& + 1.207761310x^3y^3 - 0.2685326350x^2y^4 - 0.1342718440x^4y - 0.5886496210x^3y^2 \\
& - 0.5886496210x^2y^3 - 0.1342718440xy^4 + 1.058689499x^3y - 0.5886496210x^2y^2 \\
& + 1.058689499xy^3 - 0.2685326350x^2y - 0.2685326350xy^2 + 1.058689499xy \\
& - 0.1342718440y^2 - 0.1342718440y^3 - 0.1342718440x^2 - 0.1342718440x^3 \\
& - 0.1342718440)/(x^3y^3),
\end{aligned}$$

$$\begin{aligned}
P_{22}(\omega) = & (-0.0567148109y - 0.0567148109x - 0.0567148109x^6y^6 - 0.0567148109x^6y^5 \\
& - 0.0567148109x^5y^6 - 0.0567148109x^6y^4 + 0.2958094510x^5y^5 - 0.0567148109x^4y^6 \\
& - 0.0567148109x^6y^3 - 0.1134296218x^5y^4 - 0.1134296218x^4y^5 - 0.0567148109x^3y^6 \\
& + 0.2958094510x^5y^3 - 0.1311715820x^4y^4 + 0.2958094510x^3y^5 - 0.0567148109x^5y^2 \\
& - 0.1311715820x^4y^3 - 0.1311715820x^3y^4 - 0.0567148109x^2y^5 - 0.1134296218x^4y^2 \\
& + 0.4750078230x^3y^3 - 0.1134296218x^2y^4 - 0.0567148109x^4y - 0.1311715820x^3y^2 \\
& - 0.1311715820x^2y^3 - 0.0567148109xy^4 + 0.2958094510x^3y - 0.1311715820x^2y^2 \\
& + 0.2958094510xy^3 - 0.1134296218x^2y - 0.1134296218xy^2 + 0.2958094510xy \\
& - 0.0567148109y^2 - 0.0567148109y^3 - 0.0567148109x^2 - 0.0567148109x^3 \\
& - 0.0567148109)/(x^3y^3).
\end{aligned}$$

and

$$\tilde{\mathbf{P}}(\omega) = \begin{bmatrix} \tilde{P}_{11}(\omega) & \tilde{P}_{12}(\omega) \\ \tilde{P}_{21}(\omega) & \tilde{P}_{22}(\omega) \end{bmatrix}$$

where

$$\begin{aligned}
\tilde{P}_{11}(\omega) = & (0.01719052764x^4y^4 + 0.125x^4y^3 + 0.125x^3y^4 + 0.01719052764x^4y^2 + 0.01719052764x^2y^4 \\
& + 0.125x^3y + 0.1472389204x^2y^2 + 0.125xy^3 + 0.01719052764x^2 + 0.01719052764y^2 \\
& + 0.125x + 0.125y + 0.01719052764)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_{12}(\omega) = & (-0.1018640762x^4y^4 - 0.06511976048x^4y^3 - 0.06511976048x^3y^4 - 0.1018640762x^4y^2 \\
& + 0.06511976048x^3y^3 - 0.1018640762x^2y^4 + 0.06511976048x^3y^2 + 0.06511976048x^2y^3 \\
& - 0.06511976048x^3y + 0.6109105102x^2y^2 - 0.06511976048xy^3 + 0.06511976048x^2y \\
& + 0.06511976048xy^2 - 0.1018640762x^2 + 0.06511976048xy - 0.1018640762y^2 \\
& - 0.06511976048x - 0.06511976048y - 0.1018640762)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_{21}(\omega) = & (0.01814093113x^4y^4 + 0.1548951049x^4y^3 + 0.1548951049x^3y^4 + 0.01814093113x^4y^2 \\
& + 0.01814093113x^2y^4 + 0.1548951049x^3y - 0.008133096475x^2y^2 + 0.1548951049xy^3 \\
& + 0.01814093113x^2 + 0.01814093113y^2 + 0.1548951049x + 0.1548951049y \\
& + 0.01814093113)/(x^2y^2),
\end{aligned}$$

$$\begin{aligned}\tilde{P}_{22}(\omega) = & (-0.1220459440x^4y^4 - 0.0768750x^4y^3 - 0.0768750x^3y^4 - 0.1220459440x^4y^2 \\ & + 0.09301606922x^3y^3 - 0.1220459440x^2y^4 + 0.09301606922x^3y^2 + 0.09301606922x^2y^3 \\ & - 0.0768750x^3y + 0.1733734162x^2y^2 - 0.0768750xy^3 + 0.09301606922x^2y \\ & + 0.09301606922xy^2 - 0.1220459440x^2 + 0.09301606922xy - 0.1220459440y^2 \\ & - 0.0768750x - 0.0768750y - 0.1220459440)/(x^2y^2).\end{aligned}$$

The lowpass filter $\mathbf{P}(\omega)$ supported on $[-3, 3]^2$ and the scaling function Φ corresponding to this lowpass filter is the continuous C^2 box-spline B_{333} associated with the direction sets for box-splines

$$\begin{bmatrix} 1 & 0 & -1 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 & 2 & -2 \end{bmatrix}$$

The lowpass filter $\tilde{\mathbf{P}}(\omega)$ supported on $[-2, 2]^2$ and the scaling function $\tilde{\Phi}$ corresponding to this lowpass filter is the continuous C^2 cubic box-spline B_{222} associated with the direction sets for box-splines. We call this type of framelets **Loop's scheme-based bi-framelets** denoted its multiple frame filter bank by *Loop* - $F_{3,2}$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

If we want to have smoother $\tilde{\Phi}$, we need more steps of algorithms.

The highpass analysis mask $\mathbf{Q}^{(\ell)}$ and the highpass synthesis mask $\tilde{\mathbf{Q}}^{(\ell)}$ will be as follows:

$$\mathbf{Q}_1^{(1)}(\omega) = \begin{bmatrix} Q_{11}^{(1)}(\omega) & Q_{12}^{(1)}(\omega) \\ Q_{21}^{(1)}(\omega) & Q_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}Q_{11}^{(1)}(\omega) = & (-0.2977794955 - 0.2977794955x^2 - 0.02977794955x^3 - 0.2977794955x^4y^6 \\ & - 0.059557029x^4y^5 - 0.2977794955x^3y^6 - 0.1516438790x^4y^4 + 0.2569731570x^3y^5 \\ & - 0.1516438790x^4y^3 - 0.1516438790x^3y^4 - 0.2977794955x^2y^5 - 0.05955702900x^2y^4 \\ & - 0.1516438790x^3y^2 - 0.1516438790x^2y^3 - 0.2977794955xy^4 - 0.1516438790x^2y^2 \\ & + 2.569731570xy^3 - 5.955702900x^2y - 0.059557029xy^2 + 2.569731570xy \\ & - 0.2977794955y^2 - 0.2977794955y^3 + 0.2613797670x^3y^3 - 0.2977794955x^6y^6 \\ & - 0.2977794955x^6y^5 - 0.2977794955x^5y^6 - 0.2977794955x^6y^4 + 0.2569731570x^5y^5 \\ & - 0.2977794955x^6y^3 - 0.005955702900x^5y^4 + 0.2569731570x^5y^3 - 0.2977794955x^5y^2 \\ & - 0.05955702900x^4y^2 - 0.2977794955x^4y + 2.569731570x^3y - 0.2977794955y \\ & - 2.977794955x)/(x^3y^3),\end{aligned}$$

$$\begin{aligned}
Q_{12}^{(1)}(\omega) = & (-2.186444420x^2 - 2.186444420x^3 - 21.86444420x^4y^6 - 0.4372888840x^4y^5 \\
& - 21.86444420x^3y^6 + 4.768062780x^4y^4 + 10.35244894x^3y^5 + 4.768062780x^4y^3 \\
& + 4.768062780x^3y^4 - 21.86444420x^2y^5 - 4.372888840x^2y^4 + 4.768062780x^3y^2 \\
& + 4.768062780x^2y^3 - 2.186444420xy^4 + 0.4768062780x^2y^2 + 103.5244894xy^3 \\
& - 43.72888840x^2y - 0.4372888840xy^2 + 1035.244894xy - 2.186444420y^2 \\
& - 21.86444420y^3 - 15.19532116x^3y^3 - 21.86444420x^6y^6 - 0.2186444420x^6y^5 \\
& - 21.86444420x^5y^6 - 21.86444420x^6y^4 + 1.035244894x^5y^5 - 2.186444420x^6y^3 \\
& - 4.372888840x^5y^4 + 10.35244894x^5y^3 - 21.86444420x^5y^2 - 4.372888840x^4y^2 - \\
& 21.86444420x^4y + 103.5244894x^3y - 2.186444420y - 2.186444420x \\
& - 2.186444420)/(x^3y^3),
\end{aligned}$$

$$\begin{aligned}
Q_{21}^{(1)}(\omega) = & (-0.2014984390x^2 - 0.2014984390x^3 - 0.2014984390x^4y^6 - 0.4029968780x^4y^5 \\
& - 0.2014984390x^3y^6 - 0.8923034120x^4y^4 + 0.1711773220x^3y^5 - 0.8923034120x^4y^3 \\
& - 0.8923034120x^3y^4 - 0.2014984390x^2y^5 - 0.4029968780x^2y^4 - 0.8923034120x^3y^2 \\
& - 8.923034120x^2y^3 - 2.014984390xy^4 - 0.8923034120x^2y^2 + 0.1711773220xy^3 \\
& - 0.4029968780x^2y - 0.4029968780xy^2 + 17.11773220xy - 0.2014984390y^2 \\
& - 0.2014984390y^3 + 1.128108104x^3y^3 - 0.2014984390x^6y^6 - 0.2014984390x^6y^5 \\
& - 0.2014984390x^5y^6 - 0.2014984390x^6y^4 + 0.1711773220x^5y^5 - 2.014984390x^6y^3 \\
& - 0.4029968780x^5y^4 + 0.1711773220x^5y^3 - 0.2014984390x^5y^2 - 0.4029968780x^4y^2 \\
& - 2.014984390x^4y + 0.1711773220x^3y - 0.2014984390y - 0.2014984390x \\
& - 0.2014984390)/(x^3y^3)
\end{aligned}$$

$$\begin{aligned}
Q_{22}^{(1)}(\omega) = & (-0.1366087424x^2 - 0.1366087424x^3 - 0.1366087424x^4y^6 - 0.02732206100x^4y^5 \\
& - 0.1366087424x^3y^6 - 0.04074166980x^4y^4 + 0.06541981160x^3y^5 - 0.0407416698x^4y^3 \\
& - 0.04074166980x^3y^4 - 0.1366087424x^2y^5 - 0.02732206100x^2y^4 - 0.04074166980x^3y^2 \\
& - 0.04074166980x^2y^3 - 0.1366087424xy^4 - 0.04074166980x^2y^2 + 0.06541981160xy^3 \\
& - 2.732206100x^2y - 0.02732206100xy^2 + 6.541981160xy - 0.1366087424y^2 \\
& - 0.1366087424y^3 + 0.4808432720x^3y^3 - 0.01366087424x^6y^6 - 0.1366087424x^6y^5 \\
& - 0.1366087424x^5y^6 - 0.1366087424x^6y^4 + 0.06541981160x^5y^5 - 0.1366087424x^6y^3 \\
& - 0.02732206100x^5y^4 + 0.06541981160x^5y^3 - 0.1366087424x^5y^2 - 0.02732206100x^4y^2 \\
& - 0.1366087424x^4y + 0.6541981160x^3y - 0.1366087424y - 0.1366087424x \\
& - 0.1366087424)/(x^3y^3)
\end{aligned}$$

$$\mathbf{Q}_2^{(2)}(\omega) = \begin{bmatrix} Q_{11}^{(2)}(\omega) & Q_{12}^{(2)}(\omega) \\ Q_{21}^{(2)}(\omega) & Q_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}
Q_{11}^{(2)}(\omega) &= (-0.16667x^2y^2 - 0.16667x^2y - 0.16667xy^2 - 0.16667x + 1.00002xy \\
&\quad - 0.16667y - 0.16667)/xy, \\
Q_{12}^{(2)}(\omega) &= 0, \\
Q_{21}^{(2)}(\omega) &= 0, \\
Q_{22}^{(2)}(\omega) &= (-0.4060880x^2y^2 - 0.4060880x^2y - 0.4060880xy^2 - 0.4060880x + 0.9999917xy \\
&\quad - 0.4060880y - 0.4060880)/(xy).
\end{aligned}$$

$$\mathbf{Q}_2^{(3)}(\omega) = \begin{bmatrix} Q_{11}^{(3)}(\omega) & Q_{12}^{(3)}(\omega) \\ Q_{21}^{(3)}(\omega) & Q_{22}^{(3)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}
Q_{11}^{(3)}(\omega) &= (0.01340011460x^4y^4 + 0.09502856747x^4y^2 + 0.01340011460x^4y^3 + 0.09502856747x^4y \\
&\quad + 0.01340011460x^3y^4 - 0.08040127100x^3y^3 + 0.1084278070x^3y^2 - 1.070159790x^3y \\
&\quad + 0.09502856750x^3 + 0.09502856750x^2y^4 + 0.1084278070x^2y^3 + 1.026784000x^2y^2 \\
&\quad + 0.1084278070x^2y + 0.09502856750x^2 + 0.09502856750xy^4 - 1.070159790xy^3 \\
&\quad + 0.1084278070xy^2 - 0.08040127100xy + 0.01340011460x + 0.09502856750y^3 \\
&\quad + 0.01340011460y + 0.09502856750y^2 + 0.01340011460)/(x^3y^3), \\
Q_{12}^{(3)}(\omega) &= (-0.1589304420x^4y^4 - 0.1589304420x^4y^3 + 0.251526x^4y^2 + 0.251526x^4y \\
&\quad - 0.1589304420x^3y^4 + 0.3913608600x^3y^3 + 0.09259555800x^3y^2 - 0.9469274100x^3y \\
&\quad + 0.251526x^3 + 0.251526x^2y^4 + 0.09259555796x^2y^3 - 0.3178574849x^2y^2 \\
&\quad + 0.9259555796x^2y + 0.251526x^2 + 0.251526xy^4 - 0.9469274100xy^3 \\
&\quad + 0.9259555800xy^2 + 0.3913608600xy - 0.1589304420x + 0.251526y^3 \\
&\quad + 0.2515260000y^2 - 0.1589304420y - 0.1589304420)/(x^3y^3), \\
Q_{21}^{(3)}(\omega) &= (-0.003732794520x^4y^4 - 0.003732794520x^4y^3 + 0.07549717630x^4y^2 + 0.07549717630x^4y \\
&\quad - 0.003732794520x^3y^4 - 0.2881329360x^3y^3 + 0.07176378300x^3y^2 - 0.1424527559x^3y \\
&\quad + 0.07549717630x^3 + 0.07549717630x^2y^4 + 0.07176378298x^2y^3 - 0.007465589024x^2y^2 \\
&\quad + 0.07176378299x^2y + 0.07549717629x^2 + 0.07549717630xy^4 - 0.1424527559xy^3 \\
&\quad + 0.07176378300xy^2 - 0.2881329360xy - 0.003732794520x + 0.07549717630y^3 \\
&\quad + 0.07549717630y^2 - 0.003732794520y - 0.003732794520)/(x^3y^3), \\
Q_{22}^{(3)}(\omega) &= (0.004370499595x^4y^4 + 0.004370499595x^4y^3 - 0.09423882850x^4y^2 - 0.09423882850x^4y \\
&\quad + 0.004370499595x^3y^4 - 0.2023915220x^3y^3 - 0.08986589950x^3y^2 + 0.4418866840x^3y \\
&\quad - 0.09423882850x^3 - 0.09423882850x^2y^4 - 0.08986589950x^2y^3 + 1.008733127x^2y^2 \\
&\quad - 0.08986589950x^2y - 0.09423882850x^2 - 0.09423882850xy^4 + 0.4418866840xy^3 \\
&\quad - 0.08986589950xy^2 - 0.2023915220xy + 0.004370499595x - 0.09423882850y^3 \\
&\quad - 0.09423882850y^2)/(x^3y^3).
\end{aligned}$$

$$\mathbf{Q}_2^{(4)}(\omega) = \begin{bmatrix} Q_{11}^{(4)}(\omega) & Q_{12}^{(4)}(\omega) \\ Q_{21}^{(4)}(\omega) & Q_{22}^{(4)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned} Q_{11}^{(4)}(\omega) = & \frac{1}{xy^3}(0.09502856750x^4y^6 + 0.09502856750x^3y^6 + 0.09502856750x^4y^5 - 1.070159790x^3y^5 \\ & + 0.09502856750x^2y^5 + 0.01340011460x^4y^4 + 0.1084278070x^3y^4 + 0.1084278070x^2y^4 \\ & + 0.01340011460xy^4 + 0.01340011460x^4y^3 - 0.08040127100x^3y^3 + 1.026784000x^2y^3 \\ & - 0.08040127100xy^3 + 0.01340011460y^3 + 0.01340011460x^3y^2 + 0.1084278070x^2y^2 \\ & + 0.1084278070xy^2 + 0.01340011460y^2 + 0.09502856750x^2y - 1.070159790xy \\ & + 0.09502856750y + 0.09502856750x + 0.09502856750), \end{aligned}$$

$$\begin{aligned} Q_{12}^{(4)}(\omega) = & \frac{1}{xy^3}(0.251526x^4y^6 + 0.251526x^3y^6 + 0.251526x^4y^5 - 0.9469274100x^3y^5 \\ & + 0.251526x^2y^5 - 0.1589304420x^4y^4 + 0.09259555800x^3y^4 + 0.09259555800x^2y^4 \\ & - 0.1589304420xy^4 - 0.1589304420x^4y^3 + 0.391360860x^3y^3 - 0.3178574850x^2y^3 \\ & + 0.391360860xy^3 - 0.1589304420y^3 - 0.1589304420x^3y^2 + 0.9259555800x^2y^2 \\ & + 0.09259555800xy^2 - 0.1589304420y^2 + 0.251526x^2y - 0.946927410xy \\ & + 0.251526y + 0.2515260000x + 0.2515260000), \end{aligned}$$

$$\begin{aligned} Q_{21}^{(4)}(\omega) = & \frac{1}{xy^3}(0.07549717630x^4y^6 + 0.07549717630x^3y^6 + 0.07549717630x^4y^5 - 0.1424527559x^3y^5 \\ & + 0.07549717630x^2y^5 - 0.003732794520x^4y^4 + 0.07176378300x^3y^4 + 0.07176378300x^2y^4 \\ & - 0.003732794520xy^4 - 0.003732794520x^4y^3 - 0.007465589040x^2y^3 - 0.2881329360x^3y^3 \\ & - 0.2881329360xy^3 - 0.003732794520y^3 - 0.003732794520x^3y^2 + 0.07176378300x^2y^2 \\ & + 0.0717637830xy^2 - 0.003732794520y^2 + 0.07549717630x^2 * y - 0.1424527559xy \\ & + 0.07549717630y + 0.07549717630x + 0.07549717630), \end{aligned}$$

$$\begin{aligned} Q_{22}^{(4)}(\omega) = & -\frac{1}{xy^3}(-0.09423882850x^4y^6 - 0.09423882850x^3y^6 - 0.9423882850x^4y^5 + 0.4418866840x^3y^5 \\ & - 0.9423882850x^2y^5 + 4.370499595x^4y^4 - 0.8986589950x^3y^4 - .8986589950x^2y^4 \\ & + 4.370499595xy^4 + 4.370499595x^4y^3 - 2.023915220x^3y^3 + 1.008733127x^2y^3 \\ & - 2.023915220xy^3 + 4.370499595y^3 + 4.370499595x^3y^2 - 0.8986589950x^2y^2 \\ & - 8.986589950xy^2 + 4.370499595y^2 - 0.9423882850x^2y + 44.18866840xy \\ & - 0.9423882850y - 94.23882850x - 94.23882850). \end{aligned}$$

and

$$\tilde{\mathbf{Q}}_1^{(1)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(1)}(\omega) & \tilde{Q}_{12}^{(1)}(\omega) \\ \tilde{Q}_{21}^{(1)}(\omega) & \tilde{Q}_{22}^{(1)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}\tilde{Q}_{11}^{(1)}(\omega) = & (0.1174866107x^4y^4 + 0.2011458740x^4y^3 + 0.1174866107x^4y^2 + 0.2011458740x^3y^4 \\ & + 0.02010062070x^3y^3 + 0.02010062070x^3y^2 + 0.2011458740x^3y + 0.1174866107x^2y^4 \\ & + 0.02010062070x^2y^3 + 0.3479485880x^2y^2 + 0.02010062070x^2y + 0.1174866107x^2 \\ & + 0.2011458740xy^3 + 0.02010062070xy^2 + 0.02010062070xy + 0.2011458740x \\ & + 0.1174866107y^2 + 0.2011458740y + 0.1174866107)/(x^2y^2),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{12}^{(1)}(\omega) = & (0.07915557650x^4y^4 + 0.7753104309x^4y^3 + 0.07915557650x^4y^2 + 0.07753104310x^3y^4 \\ & + 0.3206401030x^3y^3 + 0.3206401030x^3y^2 + 0.07753104310x^3y + 0.07915557650x^2y^4 \\ & + 0.3206401030x^2y^3 + 0.08317461740x^2y^2 + 0.3206401030x^2y + 0.07915557650x^2 \\ & + 0.07753104310xy^3 + 0.3206401030xy^2 + 0.3206401030xy + 0.07753104310x \\ & + 0.07915557650y^2 + 0.07753104310y + 0.07915557650)/(x^2y^2),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{21}^{(1)}(\omega) = & (0.03627152440x^4y^4 + 0.1476999320x^4y^3 + 0.03627152440x^4y^2 + 0.1476999320x^3y^4 \\ & - 0.09784153320x^3y^3 - 0.09784153320x^3y^2 + 0.1476999320x^3y + 0.03627152440x^2y^4 \\ & - 0.09784153320x^2y^3 - 0.07550717640x^2y^2 - 0.09784153320x^2y + 0.03627152440x^2 \\ & + 0.1476999320xy^3 - 0.09784153320xy^2 - 0.09784153320xy + 0.1476999320x \\ & + 0.03627152440y^2 + 0.1476999320y + 0.03627152440)/(x^2y^2),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{22}^{(1)}(\omega) = & (-0.04362858240x^4y^4 - 0.05343970559x^4y^3 - 0.04362858240x^4y^2 - 0.05343970560x^3y^4 \\ & - 0.001488057600x^3y^3 - 0.001488057600x^3y^2 - 0.05343970560x^3y - 0.04362858240x^2y^4 \\ & - 0.001488057600x^2y^3 + 0.2471528960x^2y^2 - 0.001488057600x^2y - 0.04362858240x^2 \\ & - 0.05343970560xy^3 - 0.001488057600xy^2 - 0.001488057600xy - 0.05343970560x \\ & - 0.04362858240y^2 - 0.05343970560y - 0.04362858240)/(x^2y^2).\end{aligned}$$

$$\tilde{\mathbf{Q}}_2^{(2)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(2)}(\omega) & \tilde{Q}_{12}^{(2)}(\omega) \\ \tilde{Q}_{21}^{(2)}(\omega) & \tilde{Q}_{22}^{(2)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}\tilde{Q}_{11}^{(2)}(\omega) = & \frac{1}{x^4y^4}(15.35269048x^2 + 15.48106632y^2 + 49.39760992x^4y^4 + 28.84628680x^4y^3 \\ & + 28.84628680x^3y^4 + 3.096154376x^4y^2 + 1.626663224x^3y^3 + 3.096154376x^2y^4 \\ & + 2.884628680x^3y^2 + 2.884628680x^2y^3 + 3.321342088x^3y + 49.52539688x^2y^2 \\ & + 3.321342088xy^3 - 43.66780752x^2y - 43.66780752xy^2 - 43.66780752xy \\ & + 33.21342088y + 33.21342088x + 1.548106632x^6y^6 + 33.21342088x^6y^5 \\ & + 33.21342088x^5y^6 + 15.48106632x^6y^4 - 43.66780752x^5y^5 \\ & + 15.48106632x^4y^6 - 43.66780752x^5y^4 - 43.66780752x^4y^5 \\ & + 33.21342088x^5y^3 + 33.21342088x^3y^5 + 15.48106632),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{12}^{(2)}(\omega) = & \frac{1}{x^4 y^4} (0.03941033920 + 0.03933549760x^2 + 0.03941033920y^2 + 0.03360076000x^4 y^4 \\ & + 0.07044153760x^4 y^3 + 0.07044153760x^3 y^4 + 0.07882223760x^4 y^2 + 0.1054003608x^3 y^3 \\ & + 0.07882223760x^2 y^4 + 0.07044153760x^3 y^2 + 0.07044153760x^2 y^3 + 0.01774213680x^3 y \\ & + 0.03360855600x^2 y^2 + 0.01774213680xy^3 + 0.05269940080x^2 y + 0.05269940080xy^2 \\ & + 0.05269940080xy + 0.01774213680y + 0.01774213680x + 0.00003941033920x^6 y^6 \\ & + 0.01774213680x^6 y^5 + 0.01774213680x^5 y^6 + 0.03941033920x^6 y^4 + 0.05269940080x^5 y^5 \\ & + 0.03941033920x^4 y^6 + 0.05269940080x^5 y^4 + 0.05269940080x^4 y^5 + 0.01774213680x^5 y^3 \\ & + 0.01774213680x^3 y^5),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{21}^{(2)}(\omega) = & \frac{1}{x^4 y^4} (0.1091717610x^2 + 0.1105257670y^2 + 0.3125139690x^4 y^4 + 0.2088554255x^4 y^3 \\ & + 0.2088554255x^3 y^4 + 0.2210515340x^4 y^2 - 0.06578994509x^3 y^3 + 0.2210515340x^2 y^4 \\ & + 0.2088554255x^3 y^2 + 0.2088554255x^2 y^3 + 0.2417503980x^3 y + 0.3138679750x^2 y^2 \\ & + 0.2417503980xy^3 - 0.3289497250x^2 y - 0.3289497250xy^2 - 0.3289497250xy \\ & + 0.2417503980y + 0.2417503980x + 0.001105257670x^6 y^6 + 0.2417503980x^6 y^5 \\ & + 0.2417503980x^5 y^6 + 0.1105257670x^6 y^4 - 0.3289497250x^5 y^5 + 0.1105257670x^4 y^6 \\ & - 0.3289497250x^5 y^4 - 0.3289497250x^4 y^5 + 0.2417503980x^5 y^3 + 0.2417503980x^3 y^5 \\ & + 1.105257670),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{22}^{(2)}(\omega) = & \frac{1}{x^4 y^4} (0.2042942700x^2 + 0.2050958880y^2 + 0.03504638400x^4 y^4 + 0.04400320800x^4 y^3 \\ & + 0.04400320800x^3 y^4 + 0.04101858600x^4 y^2 + 0.03373303200x^3 y^3 + 0.04101858600x^2 y^4 \\ & + 0.04400320800x^3 y^2 + 0.04400320800x^2 y^3 + 0.03373007400x^3 y + 0.03505525800x^2 y^2 \\ & + 0.03373007400xy^3 + 0.04366599600x^2 y + 0.04366599600xy^2 + 0.04366599600xy \\ & + 0.03373007400y + 0.03373007400x + 0.0000205095888x^6 y^6 + 0.03373007401x^6 y^5 \\ & + 0.03373007400x^5 y^6 + 0.2050958880x^6 y^4 + 0.04366599600x^5 y^5 + 0.2050958880x^4 y^6 \\ & + 0.04366599600x^5 y^4 + 0.04366599600x^4 y^5 + 0.03373007400x^5 y^3 + 0.03373007400x^3 y^5 \\ & + 0.2050958880).\end{aligned}$$

$$\tilde{\mathbf{Q}}_2^{(3)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(3)}(\omega) & \tilde{Q}_{12}^{(3)}(\omega) \\ \tilde{Q}_{21}^{(3)}(\omega) & \tilde{Q}_{22}^{(3)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned}\tilde{Q}_{11}^{(3)}(\omega) = & \frac{1}{x^2y^2}(0.3321342088x^3 + 0.3096154376x^2 + 0.1548106632y^2 + 0.1548106632x^4 \\ & + 0.3096154376x^4y^4 + 0.2884628680x^4y^3 + 0.3321342088x^3y^4 + 0.4939760992x^4y^2 \\ & + 0.2884628680x^3y^2 + 0.1548106632x^2y^4 + 0.1626663224x^3y^2 - 0.04366780752x^2y^3 \\ & + 0.2884628680x^3y + 0.4939760992x^2y^2 + 0.3321342088xy^3 + 0.2884628680x^2y \\ & - 0.04366780752xy^2 - 0.04366780752xy + 0.3321342088y + 0.3321342088x \\ & + 0.1548106632x^6y^4 + 0.3321342088x^5y^4 - 0.04366780752x^5y^3 + 0.1548106632 \\ & + 0.3321342089x^6y^3 + 0.1548106632x^6y^2 - 0.04366780752x^5y^2 + 0.3321342088x^5y \\ & - 0.04366780752x^4y),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{12}^{(3)}(\omega) = & \frac{1}{x^2y^2}(0.03941033920 + 0.01774213680x^3 + 0.07882223760x^2 + 0.03941033920y^2 \\ & + 0.03941033920x^4 + 0.07882223760x^4y^4 + 0.07044153760x^4y^3 + 0.01774213680x^3y^4 \\ & + 0.3360076000x^4y^2 + 0.07044153760x^3y^3 + 0.03941033920x^2y^4 + 0.1054003608x^3y^2 \\ & + 0.05269940080x^2y^3 + 0.07044153760x^3y + 0.336007600x^2y^2 + 0.01774213680xy^3 \\ & + 0.07044153760x^2y + 0.05269940080xy^2 + 0.05269940080xy + 0.01774213680y \\ & + 0.01774213680x + 0.03941033920x^6y^4 + 0.01774213680x^5y^4 + 0.05269940080x^5y^3 \\ & + 0.01774213680x^6y^3 + 0.03941033920x^6y^2 + 0.05269940080x^5y^2 + 0.01774213680x^5y \\ & + 0.05269940080x^4y),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{21}^{(3)}(\omega) = & \frac{1}{x^2y^2}(0.2417503980x^3 + 0.2210515340x^2 + 0.1105257670y^2 + 0.1105257670x^4 \\ & + 0.2210515340x^4y^4 + 0.2088554255x^4y^3 + 0.2417503980x^3y^4 + 0.3125139690x^4y^2 \\ & + 0.2088554255x^3y^3 + 0.1105257670x^2y^4 - 0.06578994509x^3y^2 - 0.03289497250x^2y^3 \\ & + 0.2088554255x^3y + 0.3125139690x^2y^2 + 0.2417503980xy^3 + 0.2088554255x^2y \\ & - 0.03289497250xy^2 - 0.03289497250xy + 0.2417503980y + 0.2417503980x \\ & + 0.1105257670x^6y^4 + 0.2417503980x^5y^4 - 0.03289497250x^5y^3 + 0.1105257670 \\ & + 0.2417503980x^6y^3 + 0.1105257670x^6y^2 - 0.03289497250x^5y^2 + 0.2417503980x^5y \\ & - 0.03289497250x^4y),\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{22}^{(3)}(\omega) = & \frac{1}{x^2y^2}(0.02050958880 + 0.0003373007400x^3 + 0.04101858600x^2 + 0.02050958880y^2 \\ & + 0.02050958880x^4 + 0.04101858600x^4y^4 + 0.04400320800x^4y^3 + 0.0003373007400x^3y^4 \\ & + 0.3504638400x^4y^2 + 0.04400320800x^3y^3 + 0.02050958880x^2y^4 + 0.3373303200x^3y^2 \\ & + 0.04366599600x^2y^3 + 0.04400320800x^3y + 0.3504638400x^2y^2 + 0.0003373007400xy^3 \\ & + 0.04400320800x^2y + 0.04366599600xy^2 + 0.04366599600xy + 0.0003373007400y \\ & + 0.0003373007400x + 0.02050958880x^6y^4 + 0.0003373007400x^5y^4).\end{aligned}$$

$$\tilde{\mathbf{Q}}_2^{(4)}(\omega) = \begin{bmatrix} \tilde{Q}_{11}^{(4)}(\omega) & \tilde{Q}_{12}^{(4)}(\omega) \\ \tilde{Q}_{21}^{(4)}(\omega) & \tilde{Q}_{22}^{(4)}(\omega) \end{bmatrix},$$

where

$$\begin{aligned} \tilde{Q}_{11}^{(4)}(\omega) = & \frac{1}{x^2 y^2} (0.3096154376y^2 + 0.3321342088y^3 + 0.1548106632x^2 + 0.1548106632y^4 \\ & + 0.2884628680x^3y^3 + 0.4939760992x^2y^4 + 0.3321342088xy^5 - 0.04366780752x^3y^2 \\ & + 0.1626663224x^2y^3 - 0.04366780752xy^4 + 0.3321342088x^3y + 0.4939760992x^2y^2 \\ & + 0.2884628680xy^3 - 0.04366780752x^2y + 0.2884628680xy^2 - 0.04366780752xy \\ & + 0.1548106632x^4y^6 + 0.3321342088x^4y^5 + 0.3321342089x^3y^6 + 0.3096154376x^4y^4 \\ & - 0.04366780752x^3y^5 + 0.1548106632x^2y^6 + 0.3321342088x^4y^3 + 0.2884628680x^3y^4 \\ & - 0.04366780752x^2y^5 + 0.1548106632x^4y^2 + 0.3321342088y + 0.3321342088x \\ & + 0.1548106632), \end{aligned}$$

$$\begin{aligned} \tilde{Q}_{12}^{(4)}(\omega) = & \frac{1}{x^2 y^2} (0.03941033920 + 0.07882223760y^2 + 0.01774213680y^3 + 0.03941033920x^2 \\ & + 0.03941033920y^4 + 0.07044153760x^3y^3 + 0.3360076000x^2y^4 + 0.01774213680xy^5 \\ & + 0.05269940080x^3y^2 + 0.1054003608x^2y^3 + 0.05269940080xy^4 + 0.01774213680x^3y \\ & + 0.3360076000x^2y^2 + 0.07044153760xy^3 + 0.05269940080x^2y + 0.07044153760xy^2 \\ & + 0.05269940080xy + 0.03941033920x^4y^6 + 0.01774213680x^4y^5 + 0.01774213680x^3y^6 \\ & + 0.07882223760x^4y^4 + 0.05269940080x^3y^5 + 0.03941033920x^2y^6 + 0.01774213680x^4y^3 \\ & + 0.07044153760x^3y^4 + 0.05269940080x^2y^5 + 0.03941033920x^4y^2 + 0.01774213680y \\ & + 0.01774213680x), \end{aligned}$$

$$\begin{aligned} \tilde{Q}_{21}^{(4)}(\omega) = & \frac{1}{x^2 y^2} (0.2210515340y^2 + 0.2417503980y^3 + 0.1105257670x^2 + 0.1105257670y^4 \\ & + 0.2088554255x^3y^3 + 0.3125139690x^2y^4 + 0.2417503980xy^5 - 0.03289497250x^3y^2 \\ & - 0.06578994509x^2y^3 - 0.03289497250xy^4 + 0.2417503980x^3y + 0.3125139690x^2y^2 \\ & + 0.2088554255xy^3 - 0.03289497250x^2y + 0.2088554255xy^2 - 0.03289497250xy \\ & + 0.1105257670x^4y^6 + 0.2417503980x^4y^5 + 0.2417503980x^3y^6 + 0.2210515340x^4y^4 \\ & - 0.03289497250x^3y^5 + 0.1105257670x^2y^6 + 0.2417503980x^4y^3 + 0.2088554255x^3y^4 \\ & - 0.03289497250x^2y^5 + 0.1105257670x^4y^2 + 0.2417503980y + 0.2417503980x \\ & + 0.1105257670), \end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{22}^{(4)}(\omega) = & \frac{1}{x^2y^2}(0.04101858600y^2 + 0.0003373007400y^3 + 0.02050958880x^2 + 0.02050958880y^4 \\
& + 0.04400320800x^3y^3 + 0.3504638400x^2y^4 + 0.0003373007400xy^5 + 0.04366599600x^3y^2 \\
& + 0.3373303200x^2y^3 + 0.04366599600xy^4 + 0.0003373007400x^3y + 0.3504638400x^2y^2 \\
& + 0.04400320800xy^3 + 0.04366599600x^2y + 0.04400320800xy^2 + 0.04366599600xy \\
& + 0.02050958880x^4y^6 + 0.0003373007400x^4y^5 + 0.0003373007401x^3y^6 + 0.04101858600x^4y^4 \\
& + 0.04366599600x^3y^5 + 0.02050958880x^2y^6 + 0.0003373007400x^4y^3 + 0.04400320800x^3y^4 \\
& + 0.04366599600x^2y^5 + 0.02050958880x^4y^2 + 0.0003373007400y + 0.0003373007400x \\
& + 0.02050958880).
\end{aligned}$$

Table 6.1: The Smoothness of biorthogonal multiwavelets/multiple bi-frames in 2-D.

Multiresolution Algorithms	Sobolev smoothness estimate for Φ	Sobolev smoothness estimate for $\tilde{\Phi}$
2-Step bi-multiwavelets	0.0074	0.4378
3-Step bi-multiwavelets	0.0107	0.8995
2-Step type II multiple bi-frames	0.5441	1.3839
3-Step type II multiple bi-frames	0.6758	2

Table 6.2: The Smoothness of biorthogonal wavelets/bi-frames in 2-D.

Multiresolution Algorithms	Sobolev smoothness estimate for Φ	Sobolev smoothness estimate for $\tilde{\Phi}$
2-Step type II bi-frames	1.5	0.4408
3-Step type II bi-frames	2	3.5

In table 6.1, the 2-Step type II multiple bi-frame algorithm are better than the algorithm in 6.2 (scalar case) which is reduced to be a biorthogonal wavelet filter bank. In table 6.2, in 3-Step type II multiple bi-frame algorithm, when ϕ is the C^2 box-spline, the highpass filter $\tilde{q}^{(1)}$ has no vanishing moment. However, in the vector case, the highpass multifilter $\tilde{\mathbf{Q}}^{(1)}$ has vanishing moment order 1.

Let $\{\mathbf{u}_{\mathbf{k}}^0\}_{\mathbf{k} \in \mathbf{Z}^2}$ be the initial input. The multiple bi-frame denoising algorithm is

$$\mathbf{L}_{\mathbf{n}} = \sum_{\mathbf{k} \in \mathbf{Z}^2} \mathbf{u}_{\mathbf{k}+\mathbf{n}}^0 \mathbf{P}_{\mathbf{k}}^T, \quad \mathbf{H}_{\mathbf{n}}^{(\ell)} = \sum_{\mathbf{k} \in \mathbf{Z}^2} \sum_{\ell=1}^L \mathbf{u}_{\mathbf{k}+\mathbf{n}}^0 \mathbf{Q}_{\mathbf{k}}^{(\ell)T}, \quad \mathbf{u}_{\mathbf{k}}^1 = \sum_{\mathbf{k} \in \mathbf{Z}^2} \mathbf{L}_{\mathbf{n}} \tilde{\mathbf{P}}_{\mathbf{k}-\mathbf{n}} + \sum_{\ell=1}^L \sum_{\mathbf{k} \in \mathbf{Z}^2} (\mathbf{H}_{\mathbf{n}}^{(\ell)}) \tilde{\mathbf{Q}}_{\mathbf{k}-\mathbf{n}}^{(\ell)} \quad (6.36)$$

where \mathbf{L} is the lowpass output and $\mathbf{H}^{(\ell)}$ are the highpass outputs. The 2-D multiple frame algorithm can be used for surface noise-removing. We show in Fig.6.8 how to remove noise

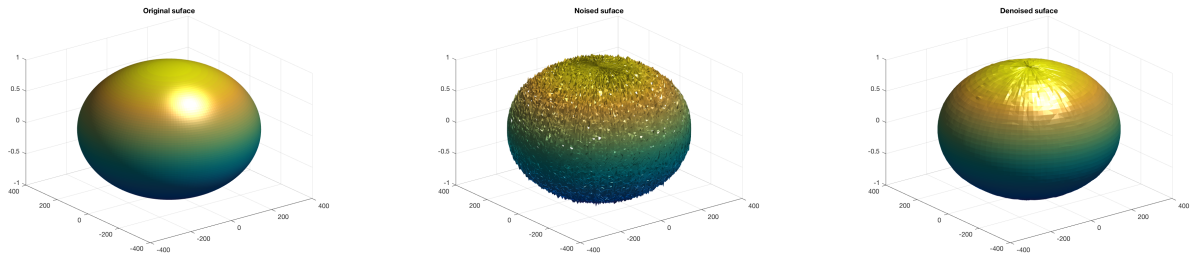


Figure 6.8: Left: original surface; middle: noised surface; right: denoised surface.

from a surface by applying 2-D denoising algorithm 4 times and soft thresholding process for noise-removing after adding Gaussian noise to the original surface. In Fig.6.8, the original surface is in the left , the noised surface is in the middle, and the denoised surface is in the right .

Chapter 7

Conclusion and Future Work

In this dissertation, we have focused on presenting highly symmetric multiple bi-frames for curve and surface multiresolution processing. We have provided the template-based method to construct multiwavelets and multiple bi-frames with each framelets being symmetric in one and two dimensions. The multiresolution algorithms in this research play an essential role on curve and surface multiresolution processing.

First, we have presented and discussed some results of one-dimensional biorthogonal multiwavelets and multiple bi-frames for curve multiresolution processing with uniform symmetry: type I and type II along with biorthogonality, sum rule orders, vanishing moments, and uniform symmetry for both types. We have presented how to construct biorthogonal multiwavelets and multiple bi-frames by using the idea of lifting scheme. Also, we have shown how to obtain the coefficients of the analysis and synthesis masks. Then, we have found the smoothness order of the analysis and synthesis multiscaling functions. In addition, we have applied our one-dimensional multiple bi-frame algorithms for curve noise-removing.

Second, we have presented some results of two-dimensional biorthogonal multiwavelets and multiple bi-frames for surface multiresolution processing with highly symmetry along with biorthogonality and 6-fold symmetry. We have shown how to construct biorthogonal multiwavelets and multiple bi-frames by using the idea of lifting scheme. Also, have presented how to obtain the analysis and synthesis masks. Then, have found the smoothness order of the analysis and synthesis multiscaling functions. Finally, we have applied our two-dimensional multiple bi-frame algorithms for surface noise-removing.

In our future work, we will investigate the construction of interpolatory biorthogonal multiwavelets and multiple bi-frames for curve and surface multiresolution processing. We will develop and study other curve and surface multiresolution applications and explore more about the algorithms that can be used in different applications besides noise-removing.

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